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DYNAMICAL COMPARISON OF MASSES

OBJECT: To investigate the effect upon the period of a vibrating system caused by varying its mass, and to make a determination of the mass of a body by a dynamical method.

METHOD: An elastic system is so arranged that its mass can be varied. The period is observed for a number of (known) masses and a curve is plotted of the mass versus the square of the period. A body of unknown mass is then added and its mass determined from observations of the period. This dynamically determined mass is then compared with the corresponding mass obtained by weighing.

THEORY: Mass is an attribute of a body that manifests itself in two ways: (1) by the tendency of the body to resist acceleration, and (2) by the gravitational attraction between the body and other bodies. Either of these properties may be employed in comparing the masses of bodies. The first aspect of mass (*inertial mass)* is expressed by Newton's second law of motion which states that the acceleration a which a body experiences is directly proportional to the force F acting upon it, inversely proportional to the mass m, and takes place in the direction of the force. The mathematical formulation of the law is

$$F = kma \qquad (1)$$

where the constant k depends upon the units used. In the absolute system k is unity. Thus the masses of bodies may be compared by determining their accelerations under the influence of equal forces. This is a dynamical method. The inertial mass of a body is therefore a measure of its tendency to resist acceleration. It depends solely upon the amount of matter which the body contains and is, therefore, independent of the position or state of motion of the body. (The theory of relativity shows the mass to depend upon the velocity, but for ordinary velocities- small compared to the velocity of light- the "relativity correction" to the mass is negligible.)

The second aspect of mass (*gravitational mass*) is represented by Newton's law of universal gravitation which states that every particle of matter in the universe attracts every other particle, the force of attraction between any two particles being directly proportional to their respective masses and inversely proportional to the square of the distance between them. Thus

$$F = G\frac{m_1 m_2}{r^2} \qquad (2)$$

where m_1 and m_2 are the masses, r is the distance between particles, and G a constant the value of which depends upon the units employed and the properties of the medium separating the bodies. While the above statement applies only to mass particles, Newton showed it to be true also for homogeneous spherical bodies where r is the distance between centers of gravity and m_1, m_2 their respective masses.

The gravitational force between a body and the earth is called the weight of the body. Weight is, therefore, a force which depends upon the gravitational mass of the body and the position of the body with respect to the earth. The gravitational masses of bodies may thus be compared directly by comparing their weights. This is a static method. Furthermore it can be shown experimentally that the ratio of the gravitational masses of two bodies is equal to the ratio of their inertial masses. Thus, if any body is arbitrarily chosen as the standard, and assigned both unit inertial mass and unit gravitational mass, the gravitational mass and the inertial mass of any body will be numerically equal when both are expressed in standard units. By international agreement the unit of gravitational and inertial mass is that of a body called the standard kilogram (a platinum-iridium cylinder) preserved at the International Bureau of Weights and Measures in Sevres, France.

The most convenient dynamical method of comparing masses involves the simple harmonic motion of a vibrating system. Simple harmonic motion is defined as motion in which the acceleration is directly proportional to the displacement and oppositely directed. Clearly this definition may be applied to either translational or rotational motion. For linear simple harmonic motion the defining equation is

$$a = -cx \qquad (3)$$

where a is the linear acceleration, x the linear displacement, and c a (dimensional) constant, the value of which depends upon the units employed and the nature of the oscillating system. The minus sign indicates that the acceleration is opposite to the displacement. A similar equation can be written in terms of angular quantities. Simple harmonic motion is therefore a type of non-uniformly accelerated motion in which the acceleration varies in a regular manner. By Newton's second law the acceleration is directly proportional to the force producing it. Hence, in the linear vibrations of material bodies under elastic or gravitational forces, the situation may be described physically by saying that the force of restitution is directly proportional to the displacement.

A simple harmonic motion is characterized by its amplitude and by either its *frequency* or its *period*. The amplitude is the maximum displacement from the rest position. The frequency is the number of vibrations per unit time, and the period is the time required for one vibration. The frequency n and the period T are thus reciprocally related, $T = l/n$. It is to be noted that the period is the time required for the complete vibration; it is, therefore, the interval between successive transits *in the same direction* through any reference point.

There exists a significant relationship between simple harmonic motion and uniform circular motion. When the motion of a point traveling with constant speed in a circular path is projected upon a straight line in the plane of the circle, the motion of the projection is seen to satisfy the above definition of simple harmonic motion. In Fig. 1 let the particle A be moving with uniform speed in a circular path of radius r. As A revolves about the center O, its projection B on the diameter of the circle oscillates to and fro with the same frequency. The amplitude of the vibration, which is the

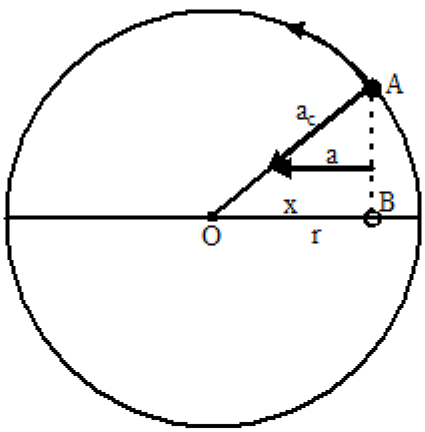

Fig. 1. Simple harmonic motion and uniform circular motion.

maximum displacement from the position of equilibrium, is equal to the radius r. The particle A has a centripetal acceleration

$$a_c = \frac{v^2}{r} = 4\pi^2 n^2 r \tag{4}$$

where n is the frequency. The component of this acceleration along the line OB is the projected linear acceleration. Thus, the acceleration a of the point B is always opposite in sign to its displacement x and varies from zero at $x = 0$ to a maximum value at $x = r$. By similar triangles

$$\frac{a}{a_c} = -\frac{x}{r} \tag{5}$$

from which

$$a = -a_c \frac{x}{r} = -4\pi^2 n^2 x \tag{6}$$

the minus sign indicating that a and x are oppositely directed. Since the frequency n is constant, this expression satisfies the definition of simple harmonic motion given by Eq. (3). Thus, any simple harmonic motion may be regarded geometrically as being generated by the uniform motion of a point in a circle whose radius is equal to the amplitude of the simple harmonic motion. Substituting $T = l/n$ in the above equation yields

$$a = -\frac{4\pi^2}{T^2} x \tag{7}$$

from which

$$T = 2\pi \sqrt{\frac{-x}{a}} \tag{8}$$

Eq. (8) is the general equation for the period of any linear simple harmonic motion. The individual expression for the period of a particular simple harmonic motion may be deduced by substituting in Eq. (8) the appropriate expression for a. Thus, in the case of an elastic vibration, the force is $f = -Kx$, where the value of the constant K depends upon the elastic properties of the body. By Newton's second law

$$a = \frac{F}{m} = \frac{-Kx}{m} \tag{9}$$

Substitution of this expression for a in Eq. (8) yields

$$T = 2\pi \frac{m}{K} \tag{10}$$

This relationship affords a means of comparing the inertias of bodies by a method which is independent of their weights.

APPARATUS: The principal piece of apparatus used in this equipment is the inertia balance illustrated in Fig. 2. The vibrating system consists of a pair of horizontal steel springs mounted on a rigid frame and connected by a platform upon which various masses can be carried. The inertia of the system can be altered by varying the added mass, and the elastic constant of the system can be adjusted by changing the length of the springs by means of the screw clamps on the frame. Since the platform is supported by two springs its motion is translational, and if the amplitude is small the motion is simple harmonic. An index rod beneath the platform provides a reference point for timing oscillations.

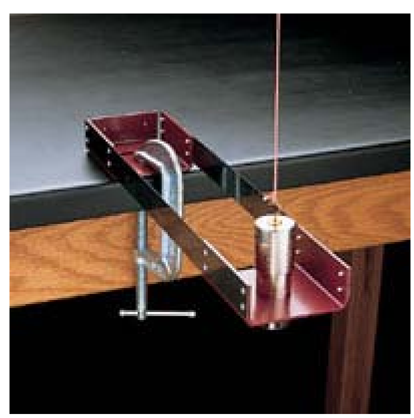
Fig. 2. The Inertia Balance

When the platform is given a slight horizontal displacement and then released it executes linear simple harmonic motion, the period of which is given by Eq. (10). In this equation the total mass m may be regarded as composed of the effective mass m_o of the platform and the added mass m_1 of the bodies on the platform. Thus

$$T = 2\pi\sqrt{\frac{m_o + m_1}{K}} \tag{10a}$$

Squaring

$$T^2 = \frac{4\pi^2}{K}\left(m_o + m_1\right) \tag{11}$$

from which

$$m_1 = CT^2 - m_o \tag{12}$$

where the constant $C = \dfrac{K}{4\pi^2}$. A graph of m_1 versus T^2 thus yields a straight line in which the slope C is proportional to

Fig. 3. The Beam Balance

the elastic constant of the system and the intercept m_o is the inherent inertia of the system when the applied mass is zero. The auxiliary apparatus consists of abeam balance such as is illustrated in Fig. 3, a set of weights, an unknown mass, and a stopwatch.

PROCEDURE:

Experimental: Determine the period of vibration for each of six values of m_1 ranging from zero to 1500 grams. Use a stopwatch and measure the time of 100 complete vibrations in each case. The amplitude of vibration should not exceed 2 or 3 centimeters and the added mass should not be more than 1000 grams. Tabulate the data and plot a curve with m_1 as the abscissa and T^2 as ordinate. This may be regarded as a calibration curve of the inertia balance. From the intercept of the curve determine the value of m_o.

Place a body of unknown mass on the platform and observe the period. By reference to the calibration curve determine the mass of the body.

Weigh the body on the beam balance and compare with the mass determined by means of the inertia balance.

Experimental Modification: The accurate determination of the period is facilitated by the use of an automatic counting device such as is illustrated in Fig. 4. A light L and a simple photoelectric cell P are so located that the beam of light falling upon the cell is intercepted by a card C attached to the vibrating platform of the inertia balance B. The photoelectric cell is connected to an amplifier and relay A which operate an impulse counter I. The relay may be arranged to operate the counter either upon the make or the break of the photoelectric current. If the apparatus is so arranged that the light is cut off from the cell only at one extreme of the swing, and if the relay operates on the break of the current, the counter will register the number of complete vibrations occurring in a given time.

QUESTIONS: 1. Show how the elastic constant of the apparatus can be determined from the calibration curve.

2. Upon what physical factors does the elastic constant depend?

3. Why is it necessary to limit the vibrations to a small amplitude?

4. Does the position of the mass m_1 on the platform affect the experiment? Explain.

5. Describe another dynamical method of comparing masses. What advantages has the vibration method over other methods?

6. If the gravitational force between two 1gm masses located 1cm apart in air is 6.67×10^{-8}dynes, and the force of attraction between the earth and a 1gm mass is 980dynes, find the mass of the earth given its radius as 6.4×10^{8}cm (about 4000 miles).

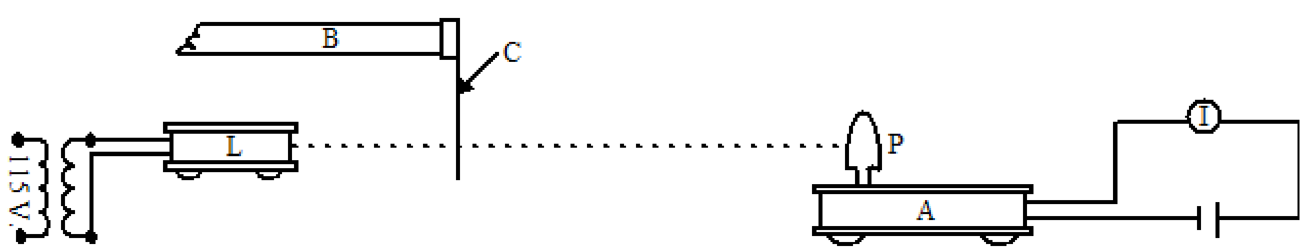

Fig. 4. Photoelectric counting device