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REFRACTION AT A SINGLE SPHERICAL SURFACE

OBJECT: To study the phenomenon of refraction at a single spherical surface.

METHOD: The phenomenon of refraction at a single spherical surface is studied with the aid of a refraction trough filled with water. The radius of curvature of the spherical window at one end of the trough is measured. As the window is thin and of uniform thickness, its refractive effect may be neglected. The spherical refracting water surface will form a real image in water of an object in air at some distance from the surface. And, likewise, if the object is placed in the trough, the refracting surface will produce a real image in air. The distances of these objects and images from the refracting surface are measured. From these pairs of distances and the radius of curvature, the relation between these quantities and the focal lengths of the single refracting surface can be verified.

THEORY: The axis of a spherical refracting surface with respect to a point M is the straight line joining M with the

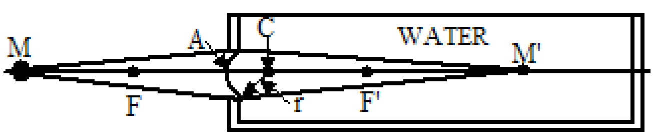

Fig. 1. Section through the Refraction Trough

center of curvature C (Fig. 1). The point A where this line cuts the refracting surface is called the *vertex* of the surface. The *absolute refractive index* n_s of a substance is defined as the ratio of the speed v with which light travels in a vacuum to the speed v_s with which it travels in the substance,

$$n_s = v / v_s \tag{1}$$

The *relative refractive index* of two substances, 1 and 2, is defined as the ratio of the light speeds v_1 and v_2 on the two sides of the boundary,

$$n_{12} = v_1 / v_2 \tag{2}$$

Two points M and M' so situated that an object placed at one point M has an image formed by the spherical refracting surface at the other point M' are called *conjugate points*. The respective distances of conjugate points from the vertex of the spherical refracting surface are called conjugate distances.

Convention of Signs: In deriving the equation for refraction at a single spherical surface, the following notation will be used:

u_A = conjugate distance measured in air
u_W = conjugate distance measured in water
n_A = absolute refractive index of air
n_W = absolute refractive index of water
r = radius of curvature of the refracting surface

All these quantities, and others, must be measured from some origin; but in general there is no one origin which is suitable for the measurement of them all. Moreover, if these quantities are to be employed in describing algebraically the nature and position of the image, care must be taken in specifying not only the origins but also the directions in which they are to be measured. For the sake of consistency and convenience, it is customary to adopt a set of conventions in the choice of origins and signs. Since the convenience afforded by anyone convention varies with the nature of the problem to which it is applied, numerous conventions are employed. For the problem at hand, the following set of conventions will be used.

Distances measured in the direction of propagation of the light are considered as positive and those measured opposite to the direction in which the light is traveling are called negative. In this treatment the light is assumed to travel from left to right.

The vertex A of the refracting surface is selected as the point of reference for measurement of the radius of curvature and the conjugate (object and image) distances. Focal lengths are measured, however, from the respective focal points. These latter quantities are defined later.

The Sagittal Formula: The curvature of a wave front or a surface at any point is equal to the reciprocal of the radius of curvature at that point. It can be shown that the curvature is proportional to a quantity which is called the sagitta of arc. This quantity can be defined by reference to Fig. 2, in which O is the center for a circle of radius r. A half chord y is dropped to the diameter AC from a point B near A, cutting off a small segment x. Since the triangles ABD and BCD are similar,

$$x / y = y / (2r - x) \quad \text{or} \quad 2rx - x^2 = y^2 \tag{3}$$

If x is small compared with the radius of the curve, x^2 may be neglected in comparison with $2rx$ and

$$x = \left(y^2 / 2\right)\left(1 / r\right) \tag{4}$$

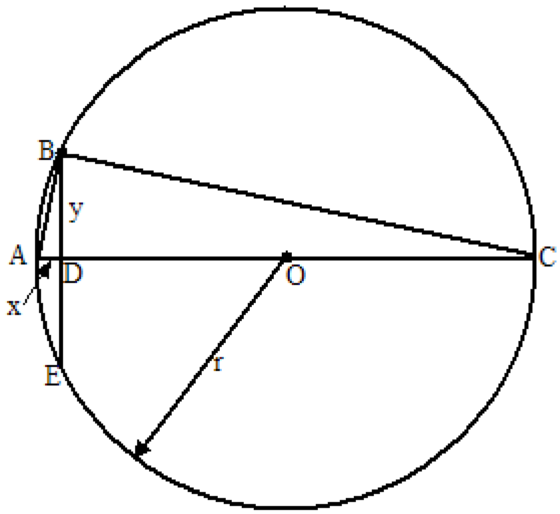

Fig. 2. AD = x is the sagitta of the arc BAE.

Hence, for any given half chord y, the curvature is proportional to the sagitta of arc x, provided x is small.

Equation of Refraction: It can be shown either experimentally or theoretically from the fundamental laws of geometrical optics that images can be formed only under certain highly restrictive conditions. Although the nature of the conditions cannot be discussed here, it is well to note that they are too stringent to be completely satisfied in practical applications of optical principles. This means that the laws of image formation which can be simply derived are likely to be conclusions drawn from a number of approximations. For many practical purposes, the simple approximations are entirely sufficient; but this characteristic of geometrical optics should be observed in derivations such as the following.

Let M in Fig. 3 be a point source from which spherical waves are sent out traveling with a speed VA. The arc ZAZ' represents a section of the refracting surface. Its radius of curvature is r, and its center of curvature is C. At the instant the wave from M strikes the surface at A, a section of the wave front is represented by the arc DAE. The radius of curvature of this wave is u_A. Since the speed v_W of the wave in water is less than the speed v_A in air, the center of the wave at A is the first to suffer retardation.

The rays drawn in Fig. 3 emanate from a point source and converge to a point image. For an object and image of finite size a series of such diagrams are required to describe completely conjugate focal relations. Furthermore, in the analysis to follow, if the aperture 2GA is small so that the ray MG is nearly parallel to the axis MM', a simple mathematical relation results which may be used in the study of image formation. Under this assumption, it can be seen from Fig. 3 that DG is approximately equal to DB, and by the time the wave front DAE has entirely passed the boundary between the air and the water, its curvature is approximately shown by the curvature BJ. This new wave front has a radius of curvature u_W and converges toward the center of curvature or image point M'.

From D draw a perpendicular to the axis intersecting it at the point H. Draw also a parallel to the axis through D cutting the surface ZZ' in B. As the edge ray MG gets nearer the axis, the distance DG becomes more nearly equal to DB and the conclusions drawn hold only when this difference may be neglected. Drop a perpendicular to the axis from B and represent the half chord BK by h.

Since the elements in a wave front are all in the same phase, the time required for the edge of the wave to go from D to B must be the same as the time required for the center of the wave to go from A to J. This time may be expressed as follows:

$$DB/v_A = AJ/v_W \qquad \text{or} \qquad n_A \cdot DB = n_W \cdot AJ \tag{5}$$

From Fig. 3,

$$DB = HK = HA + AK \quad \text{and} \quad AJ = AK - JK \tag{6}$$

Substituting in Eq. (3),

$$n_A(HA + AK) = n_W(AK - JK) \tag{7}$$

The quantities in the parentheses of Eq. (7) are respectively the sagittas of the several arcs shown in Fig. 3. Thus Eq. (4) may be applied in. order to each term of Eq. (7). Of these, HA is negative since AM = u_A is measured in a direction contrary to that in which the light is propagated. That is,

$$HA = -(h^2/2)(1/u_A) \tag{8}$$

The other terms, however, are positive. For example,

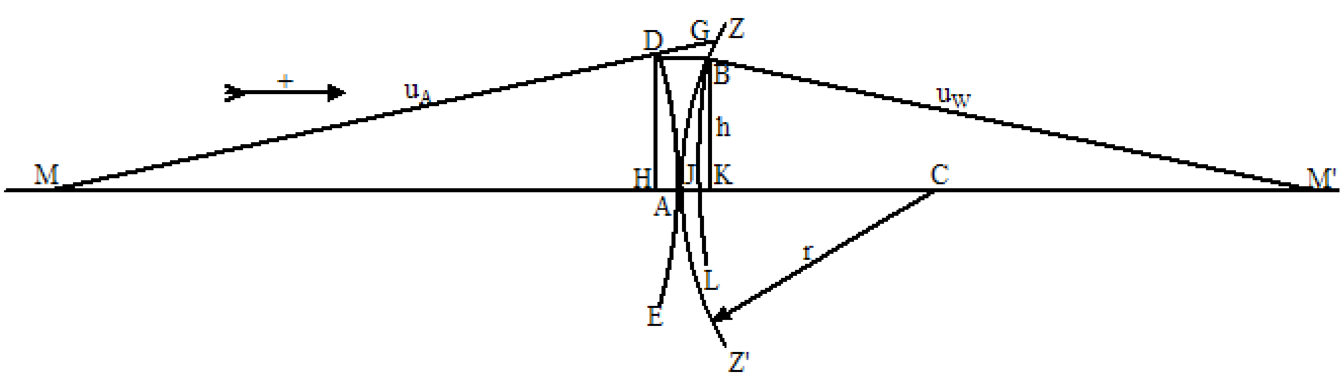

Fig. 3. Refraction at a single spherical surface.

$$AK = \left(h^2/2\right)\left(1/r\right) \quad (9)$$

Hence Eq. (7) becomes

$$-n_A h^2/2u_A + n_A h^2/2r = n_W h^2/2r - n_W h^2/2u_W \quad (10)$$

Canceling $h^2/2$ in each term, there results

$$n_W/u_W - n_A/u_A = \left(n_W - n_A\right)/r \quad (11)$$

the equation for refraction at a single spherical surface. This equation is interesting in several respects, and important because of the use which can be made of it. It was derived by considering two rays, one of which coincided with the optic axis. If a greater number of rays had been discussed, it would have been discovered that all these rays originating from the same point source would not unite at a single image point. However, if the rays considered are close to the axis, they form an image point to a very close approximation. The nature of the approximation can be expressed mathematically by considering ϕ, the angle between a given ray and the axis. The sine of this angle can be expressed in the form of a power series as

$$\sin\phi = \phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \frac{\phi^7}{7!} + ... \quad (12)$$

The mathematical physicist Gauss showed that the approximation introduced by treating optical problems in the manner employed for the derivation of Eq. (11) consists in neglecting all terms of the series expansion except the first. The method of geometrical optics in which this approximation is applied is called the Gauss first order theory. A more elaborate method is called the third order theory. Rays sufficiently close to the optic axis to make it justifiable to write $\sin\phi = \phi$ are called paraxial rays. For practical purposes, the first order theory is suitable for locating the image formed by any system of surfaces, however complex the system may be, provided that the surfaces are spherical and have their centers of curvature upon a straight line. This means that Eq. (11) is one of the most important relations in geometrical optics. By using it twice, it is comparatively easy to derive the formulas for thin lenses. Similarly, formulas for systems of lenses may be found. A plane surface is treated as a spherical surface for which *r* is infinitely large. Equation (11) may be applied to reflecting surfaces also by introducing the concept of a negative index of refraction. Since the refractive index of a substance is the ratio of the velocity of light in a vacuum to its velocity in the substance, it is apparent that negative values of refractive index indicate nothing more than such a change in direction of the velocity as occurs upon reflection.

Foci and Focal Lengths: Of the infinite number of pairs of conjugate points on the axis satisfying the relations of Eq. (11), two have special significance. The axial point on the side of the incident light whose conjugate point is at infinity is called the anterior or first principal focal point F. The axial point which is conjugate to a point at infinity on the side of the incident light is called the posterior or second principal focal point F'. The focal lengths are the respective distances of these points from the vertex of the refracting surface.

By applying these definitions to Eq. (11), the anterior focal length *f* and the posterior focal length *f'* may be obtained. Thus, if u_A' is the distance conjugate to $u_W = \infty$, it follows that

$$f = -u_A' \quad (13)$$

Likewise, if u_W' is the distance conjugate to $u_A = -\infty$,

$$f' = -u_W' \quad (14)$$

Thus, when $u_W = \infty$, Eq. (11) yields

$$n_A/f = \left(n_W - n_A\right)/r \quad (15)$$

and when $u_A = -\infty$, it follows that

$$-n_W/f' = \left(n_W - n_A\right)/r \quad (16)$$

From these last two relations, it appears that

$$-f'/f = n_W/n_A = n_{AW} \quad (17)$$

The ratio of the second focal length to the first is thus equal to the relative refractive index of the two media.

Radius of Curvature: The radius of curvature of the spherical refracting surface is to be determined by using a spherometer.

If *S* denotes the average distance between the points of the spherometer legs and *a* the difference in reading of the spherometer when all four points touch a plane surface, and when these points touch the spherical surface, then the radius of curvature *r* of the spherical surface is given by the equation

Fig. 4. Refraction Trough with accessories.

$$r = S^2/6a + a/2 \quad (18)$$

APPARATUS: The refraction trough to be used in th1s experiment (Fig. 4) consists of along glass trough w1th a thin spherical window of uniform thickness set in one end. When the trough is filled with water, the surface of the liquid next to the window will be convex outwards. As the window is very thin, its refraction may be neglected. Light entering the trough from the outside through the window may be considered as being refracted at a single spherical surface whose refractive index is that of water.

The following apparatus is needed: Refraction trough on base; large beaker for filling the trough; object holder with 40 watt lamp on base; access to 110-volt circuit; wire gauze 1½" x 2"; white screen; black screen; hooded image screen; meter stick; spherometer and plate glass; steel rule.

PROCEDURE:

Experimental: Set the spherometer on a plane glass plate and adjust the micrometer screw until all four legs touch the plane simultaneously. Read the scale and wheel setting. Repeat the measurement at least twice. Next, set the spherometer on the convex surface at the end of the trough and adjust the screw until all four legs again make contact. Then read the scale and wheel. Repeat as before. The difference between the average readings in the two cases is the distance *a* in Eq. (18).

Place a piece of smooth paper on the glass plate and press the points of the spherometer legs against this paper so that these points leave fine punctures in the paper. Use a good steel rule to measure the distance *S* between each pair of punctures. Take three independent sets of readings using anew set of punctures for each.

Object in Air, Image in Water: Place apiece of wire gauze in an object holder and mount it on a stand at the same height as the center of the spherical surface in the refraction trough and in line with the length axis of the trough. With an electric lamp illuminate the wire gauze. Fill the refraction tank with water. Place a white screen in the carrier of the trough and, moving it slowly away from the spherical surface, locate the image of the wire gauze which will be formed in the water. Measure the distance u_A from the object to the spherical surface, and the image distance u_W from the apex of the spherical surface to the white screen. Make three sets of readings for u_A and u_W. For example, make u_A = 100, 85 and 70cm and determine the corresponding value of u_W.

Object in Water, Image in Air: Remove the white image screen from the carrier and put in its place the wire gauze to serve as an object screen in the liquid. Illuminate this screen by means of the electric lamp directly back of the wire gauze. Make the distance of the object screen from the spherical end of the trough large enough so that a real image will be formed in the air. Locate this image accurately, by using the ground glass screen. Make three sets of readings. For example, make u_W = 70, 60 and 50cm and locate the three corresponding images. Note that a shadow of the wire gauze object will be formed. This must not be confused with the image of the wire gauze. The image will be formed in a small circle of high intensity.

Measure carefully for each setting the distance u_W from the object to the apex of the spherical surface and the distance u_A from the image to the spherical surface in the air. Make a fourth setting of u_A and u_W by placing the object screen at one of the end positions occupied by the image when the object was in air. Locate the image in air and record the values of u_A and u_W. Test the conjugate distances by comparing the two sets of values.

Interpretation of Data: Calculate the radius of curvature r of the spherical refracting surface by using Eq. (18). Note that when the object is in air, u_A is negative but u_W and *r* are positive. However, when the object is in water, u_A is positive but u_W and *r* are negative.

From each set of values of u_A, u_W and *r* calculate the index of refraction for water. Assume n_A = 1 for air. Determine the principal focal distances *f* and *f'* by using the average values of n_W in Eqs. (15) and (16).

QUESTIONS: 1. Locate the image of an object in air 60cm away from a single refracting surface, radius of curvature 10cm and relative refractive index *n* = 1.5.

2. If a single refracting surface separating air and water has a radius of curvature of 8cm, find the location of the object in air such that the image in water will be real and equally distant from the refracting surface.

3. Given relative refractive index *n* = 1.52 for a single refracting surface and *r* = 6cm, calculate the focal lengths *f* and *f'*.

4. Take the index of refraction of air as 1. (a) Write the form of Eq. (11) which is applicable to each surface of a simple lens, being careful to apply the convention of signs correctly. (b) Show that adding these two equations yields the lens formula

$$\frac{1}{u_A} - \frac{1}{u_A{}'} = (n-1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

where r_1 and r_2 are the radii of curvature of the two surfaces of the lens, *n* is the refractive index of the lens, u_A is the distance of the object from the first surface of the lens and u_A' is the distance of the image from the second surface.