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# THE SIMPLE HARMONIC MOTION OF A SPRING

**OBJECT**: To test Hooke's law for a spring and to observe how the period of vibration of the spring depends on the load attached to the lower end.

**METHOD**: A spring is mounted vertically and the elongation of the spring for different loads is observed. From this the force constant of the spring is obtained. With a known load attached to the spring it is further elongated and then released, thus setting the system into simple harmonic motion. The period of this motion depends on the load and the force constant of the spring.

**THEORY**: When a properly constructed spring is stretched by an applied force, it is found that the elongation of the spring is proportional to the stretching force so long as the elastic limit of the spring is not exceeded. This is Hooke's law for a spring.
If a force *F* applied to the spring produces an elongation *x*, doubling the force doubles the elongation so that the force is proportional to the elongation, or

$$F = Cx \qquad (1)$$

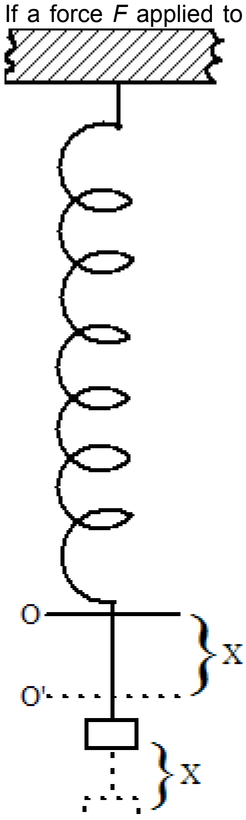

Fig. 1. Displacement of spring a vertical distance x.

where C is called the force constant of the spring and is numerically equal to the force required to produce unit elongation. Consider the spring shown in Fig. 1 which has a load of *M* grams attached at the lower end causing the spring to be stretched to the position *O*. Suppose the load is pulled down so that the spring is elongated a further distance *x* to position *O*'. The force necessary to produce this elongation is *Cx*. If the lower end of the spring is released when in the extended position, the restoring force produces an acceleration $\alpha$ in the mass as given by Newton's second law:

$$-Cx = M'a$$

or

$$a = \frac{-Cx}{M'} \qquad (2)$$

The force *Cx* is in the opposite direction to the displacement *x* as shown by the negative sign. The mass *M*' acted on by this force is not only that of the load *M* plus that of the scale

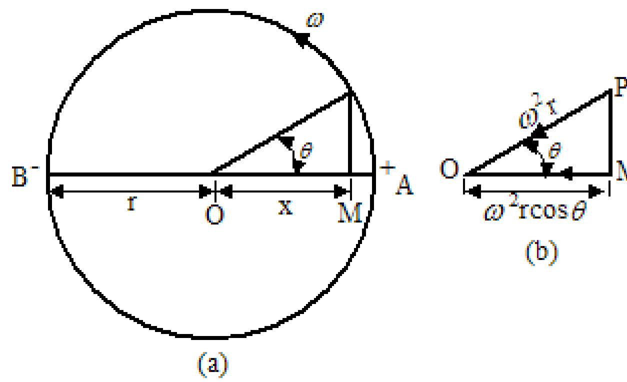

Fig. 2. (a) Diagram of point P moving with constant angular velocity $\omega$ in a circle.
(b) Vector diagram of acceleration of point P.

pan $M_p$, but also depends on the mass of the spring $M_s$.
Eq. (2) above indicates that the acceleration of the mass *M*' for an elongation *x* of the spring is proportional to the elongation *x* and is in the opposite direction. This is the condition necessary for the system to execute simple harmonic motion (S.H.M.). The period of vibration of this system may be obtained by considering another example.
*Period of Simple Harmonic Vibration*: A body executes simple harmonic vibrations whenever the acceleration of the body due to a displacement *x* is proportional to and in the opposite direction to the displacement *x*, that is, the condition necessary for S.H.M. to exist is

$$a = -kx \qquad (3)$$

where *k* is a constant and is the factor of proportionality.
The period of a S.H.M. may be found by considering a circular motion since the projection of a uniform circular motion on a diameter is simple harmonic. Consider a point p moving in a circle of radius *r* with a constant angular velocity $\omega$, Fig. 2. The acceleration of P is $\omega^2 r$ in the direction PO or $-\omega^2 r$ in the direction OP. The projection of this acceleration on the diameter AB, *in the direction OM*, is

$$a = -\omega^2 r\cos\theta = -\omega^2 \times OM$$

*Or, in the positive direction of x*

$$a = -\omega^2 x \qquad (4)$$

 

Since $\omega^2$ is constant it follows that the projection of a uniform circular motion along a diameter is a simple harmonic motion. The period *T* of this S.H.M. is the period of the point p moving in the circle or $2\pi/\omega$. Thus when the S.H.M. is given by Eq. (4) the period is

$$T = 2\pi/\omega \tag{5}$$

in general, the period of a S.H.M. is equal to 2π divided by the square root of the constant of proportionality in the equation relating the acceleration to the displacement. If

$$a = -kx \tag{6}$$

$$T = 2\pi/\sqrt{k} \tag{7}$$

*Application to S.H.M. of spring*: The motion of the spring is given by Eq. (2), hence the period of this S.H.M. is

$$T = 2\pi\sqrt{\frac{M'}{C}} \tag{8}$$

Fig. 3. Spring and attachments for studying simple harmonic motion.

**APPARATUS**: The apparatus consists of a sensitive spring attached at its upper end to a movable calibrated pillar and carrying an index mark and scale pan at its lower end as shown in Fig. 3. The upper end may be moved by a screw at the base of the pillar and the position read by means of a scale and vernier. The index marks for indicating the position of the lower end of the spring are three horizontal lines marked on a small aluminum cylinder which hangs freely within a glass tube on which a horizontal line is etched. Stops are provided at either end of the cylinder so that its motion is limited. A spring is provided as well as a set of weights, a scale pan, a balance and a timing device.

**PROCEDURE**:

**Adjustments**: Level the instrument by the leveling screws on the base so that the spring is parallel to the pillar and the aluminum cylinder hangs concentrically within the glass tube. Bring the zero point of the scale on the pillar into coincidence with the zero point of the vernier. With the scale pan in position at the lower end of the spring, move the clamp and glass tube until the etched line on the glass cylinder is in the same plane as the central index mark on the small aluminum cylinder.

**Experimental**:

***Part A***. Place a suitable weight* in the scale pan and move the upper end of the spring upwards until the etched line is in the same plane as the central index mark on the small cylinder. Record the scale and vernier readings. Repeat these observations increasing the weight in the scale pan each time by about one-tenth the maximum allowable until the maximum is reached. Remove the lower end of the spring from the small cylinder. Take the scale pan off the small aluminum cylinder and determine with the balance the mass of the pan, $M_p$. The mass of the spring *M*, should be obtained either by weighing on the balance or from the instructor.

*This weight should be about one-tenth of the maximum allowable, the latter value being given by the instructor.

***Part B***. The clamp and glass tube are moved sidewise out of line with the spring. Place the scale pan on the end of the spring. With a mass of about one-fifth the maximum allowable placed in the scale pan, pull it down vertically a small distance and then release it, being careful to avoid any sidewise motion. Measure the period of the simple harmonic vibrations set up by timing fifty or a hundred complete vibrations. Repeat the observations increasing the mass in the scale pan each time by about one-fifth the maximum allowable until the maximum is reached. Record the data in the tabular form as shown:

| Mass in scale pan in grams | Number of vibrations | Time taken in seconds | Period T in seconds | $T^2$ |
|---|---|---|---|---|
| 5 | 100 | 51.8 | 0.518 | 0.269 |
| | | | | |

**Analysis and Calculations**: Plot the graph of the data of scale readings against weights in scale pan obtained in **Part A**. A graph of such data is shown in graph (a) of Fig. 4. From the slope of the straight line the value of *C*, the force required to produce unit elongation, is obtained. In the illustration given the value of *C* is 1028 dynes per cm elongation.

Plot the graph of the data of period squared $T^2$ against load in scale pan as obtained in **Part B**. From Eq. (8) the period of vibration of the spring is

$$T = 2\pi\sqrt{\frac{M'}{C}} \quad \text{or} \quad T^2 = \frac{4\pi^2 M'}{C}$$

where *M'* is the effective mass moved by the spring and *C* is the force constant of the spring obtained above. Now *M'* is the sum of the masses of the load *M* in the scale pan, the scale pan itself $M_p$ and some fraction *f* of the mass of the spring *M*,

$$M' = M + M_p + fM_s \tag{10}$$

Hence

$$T^2 = \frac{4\pi^2}{C}\left(M + M_p + fM_s\right) \tag{11}$$

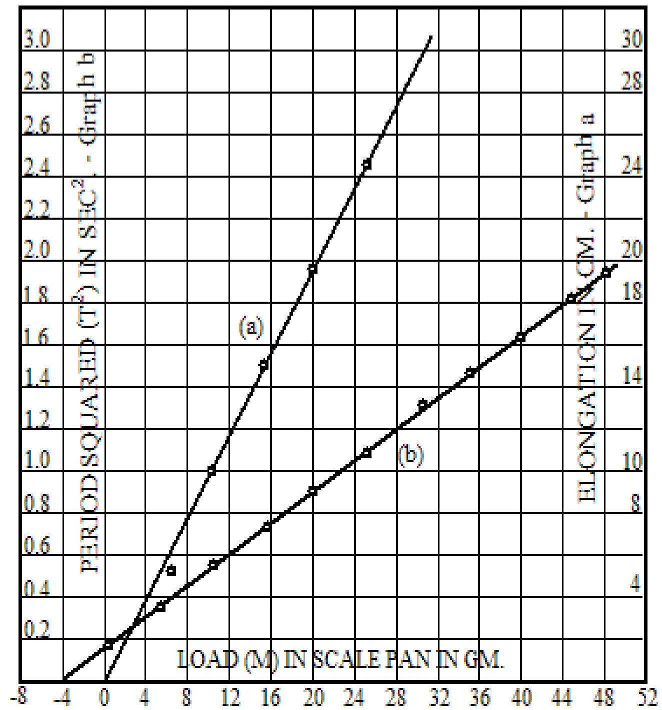


Fig. 4. Graph (a) Elongation plotted against load.
Graph (b) (period)$^2$ plotted against load.

*The slope of the line in graph (a) is*

$$\frac{27.2-0.5}{28}=\frac{26.7}{28}=0.954 \text{ cm per gm-wt}$$

*Hence the force in dynes required to produce an elongation of 1cm is*

$$C=\frac{980}{0.954}=1028 \text{ dynes per cm elongation}$$

*The slope of the line in graph (b) is*

$$\frac{1.93-0.20}{48}=\frac{1.73}{48}=0.036$$

*which from Eq.* (11) *is equal to* $4\pi^2/C$. *Hence*

$$C=\frac{4\pi^2}{0.036}=1098 \text{ dynes per cm elongation}$$

The graph of $T^2$ plotted against *M* should give a straight line as shown in graph (b), Fig. 4. The slope of the graph equals $4\pi^2/C$ and from the measured value of the slope the value of *C* may be obtained. In the illustration given *C* = 1096 dynes per cm elongation. From Eq. (11) given above it follows that $T^2$ equals zero when $M = -(M_p + fM_s)$. Hence the negative intercept *OA* on the mass axis (graph b, Fig. 4) gives the value of $M = -(M_p + fM_s)$. Using the measured values of $M_p$ and $M_s$, the fraction *f* may be obtained. From the graph of $T^2$ against load find the value of *C* and *f*.
Indicate the percentage difference between the two values found for *C*.

**QUESTIONS**: 1. Explain why the negative intercept *OA* in Fig. 4 represents the mass of the scale pan and the fractional mass of the spring.

2. If a 1 per cent error is made in the determination of the period of oscillation of the spring, what percentage error does this introduce in the calculation, from Eq. (9), of the force constant *C* of the spring?

3. If a 1 per cent error is made in the mass *M'* associated with the vibrating spring, what error would this introduce in the calculation, from Eq. (8), of (a) *C* supposing *T* is accurately known; (b) *T* supposing *C* is accurately known?

4. Show that both members of Eq. (8) have the same dimensions.