• C.J. Overbeck Northwestern University • R.R. Palmer Beloit College • Marsh W. White The Pennsylvania State University

SIMPLE HARMONIC MOTION – THE SIMPLE PENDULUM

OBJECT: To study simple harmonic motion and to measure the acceleration due to gravity with the simple pendulum.

METHOD: The period of a simple pendulum is measured at different amplitudes of vibration and for several different pendulum lengths. From these data it is demonstrated that the amplitude of a simple pendulum must be small if it is to execute simple harmonic motion, and the acceleration due to gravity is calculated from the same data.

THEORY: Simple harmonic motion is a periodic motion that can be widely observed in nature. When an object that is suspended in the earth's gravitational field is displaced from its normal position and released it oscillates to and fro with a vibratory motion about its normal position. When an elastic body is distorted and then released it too vibrates. Each of these motions can be analyzed in terms of an idealized motion known as *simple harmonic motion*. This type of motion is produced by varying forces, and hence the body experiences varying accelerations. A convenient laboratory example of simple harmonic motion is the motion of a simple pendulum.

All harmonic or periodic motions involve forces which vary with time while they act upon an object. Thus the object experiences a varying acceleration. If the acceleration of the object is at all times proportional to its displacement from the center of oscillation and is directed toward the center the motion is, by definition, *simple harmonic motion*. This definition can be expressed analytically by the equation

$$a = -cx \qquad (1)$$

where *x* is the displacement and *c* is a positive constant. The negative sign is used conventionally to show that the acceleration is opposite in sense to the displacement; that is, when the displacement is positive, the acceleration is negative; when the displacement is negative, the acceleration is positive.

Consider the idealized case of a particle of mass m that executes simple harmonic motion between two points, such as points *Q* and *Q*' in Fig. 1. The particle may be a small block that has massless horizontal springs attached to it and that rests on a horizontal frictionless surface; or, it may be a particle on the end of the prong of a vibrating tuning fork. For purposes of analysis it is not necessary to specify the nature of the forces that cause the motion; it is sufficient to know that the acceleration at each instant is given by Eq. (1). In general, two quantities characterize the motion- namely the amplitude and the period. The *amplitude* of the vibratory motion of a particle is defined as the maximum displacement of the particle from its equilibrium position. The *period* is the time required for the particle to execute one complete vibration; that is, the time between successive transits *in the same direction* through any point. The *frequency* of the vibration is defined as the reciprocal of the period. For example, if the period is 0.050 sec, the frequency is 20 vibrations per sec (20 hertz).

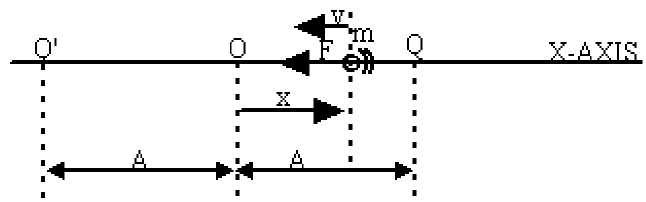

Fig. 1. A particle of mass m oscillates with simple harmonic motion between points Q and Q', with amplitude A. The force on the particle is always directed toward the center of vibration and is proportional to the displacement x. At the instant shown the particle is moving toward the left with velocity v.

Since by Newton's second law, *F = ma*, the resultant force acting on a mass *m* is proportional to its acceleration and since, by Eq. (1), for simple harmonic motion the acceleration is proportional to the displacement, the force must also be proportional to the displacement. Thus

$$F = -kx \qquad (2)$$

where *k* is a positive constant called the force constant for the motion. By equating - *kx* to *ma* and solving for the acceleration, we obtain

$$a = -(k/m)x$$

This may be written in calculus notation as

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x \qquad (3)$$

If this equation is solved for *x* as a function of time one can obtain an expression for the period *T* of the oscillation in terms of *k* and *m*.

A solution of Eq. (3) requires that the second derivative of *x* be proportional to the negative of *x*. Either the sine or the cosine of some function of time will satisfy this requirement. For, if

$$x = A\sin\omega t \qquad (4)$$

(where ω is an angular velocity in radians per sec), then by taking the first and then the second derivatives of Eq. (4), we arrive at an equation for the acceleration

$$\frac{d^2x}{dt^2} = -\omega^2 A \sin \omega t$$

By comparing this equation with Eq. (4) we see that the second derivative is proportional to the negative of the displacement if ω is a constant. From a comparison of this equation with Eq. (3) it is apparent that ω is a constant, for

$$\omega^2 = (k/m) \qquad (5)$$

The period *T* of the vibration is related to ω, as can be seen by noting that each time the argument ωt in Eq. (4) increases by 2π the sine of the new angle has the same value and, therefore, *x* has the same value as before. This will occur when the time has changed by one period, or when $\omega t + 2\pi = \omega(t+T)$. From this it follows that $T = 2\pi/\omega$.

Finally, from Eq. (5), the period is given by the equation

$$T = 2\pi\sqrt{m/k} \qquad (6)$$

This equation gives the period for the simple harmonic motion of any object of mass *m* that is subject to a force constant *k*; that is, when the restoring force is proportional to the displacement. Note finally that the constant *A* in Eq. (4) is the amplitude of the motion- the maximum value of the displacement.

A most convenient laboratory example of a simple harmonic motion device is the simple pendulum. Ideally, a simple pendulum consists of a particle that is supported by a long weightless string and that is allowed to swing to and fro in a minute arc in the earth's gravitational field. A small metal ball supported by a string whose mass is negligible relative to that of the ball and whose length is very much greater than the radius of the ball approximates these qualifications.

Consider the simple pendulum depicted in Fig. 2. A ball of mass *m* is supported from point B by a light string of length *l*. Note that *l* is the distance from the point of support to the center of the ball. When the ball is at point O, directly below B, the upward force of the string exactly balances the weight of the ball, *mg*. When the ball has the displacement *x* from this position the pull of the string and the pull of gravity do not balance. The resultant of these two forces is then $mg\sin\theta$, as is apparent from the vector diagram in Fig. 2, where θ is the angle the string makes with the vertical at the instant shown. Note that if angle θ is small, $\sin\theta$ is very nearly equal to x/l, where *x* is the displacement along the arc. The resultant force on m is then given by the equation

$$F = -\frac{mg}{l}x$$

Thus, if the amplitude of vibration is small the resultant force on the ball is at all times proportional to the displacement *x* as is required for simple harmonic motion.

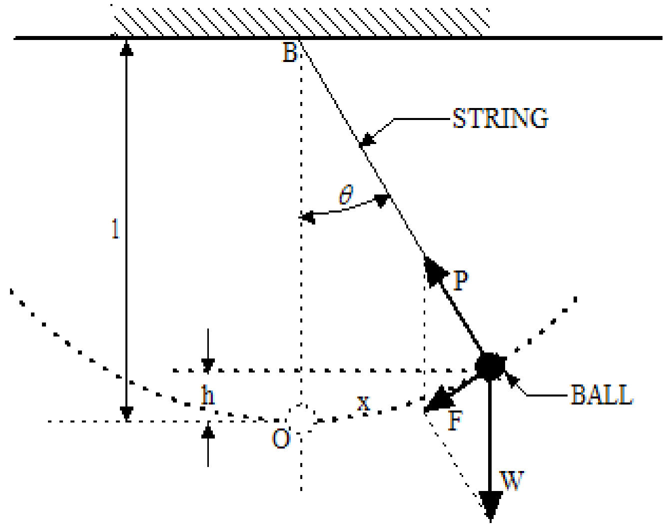

Fig. 2. Analysis of the forces acting on a ball suspended as a simple pendulum by a string of length l. At the instant shown the ball is at distance x along the arc of its path and is at height h above its lowest point.

Furthermore, the force constant *k* for the motion is seen to be *mg/l*. When this expression for *k* is substituted in Eq. (6) the period of vibration is found to be given by the equation

$$T = 2\pi\sqrt{\frac{l}{g}} \qquad (7)$$

A measurement of *T* for a given value of *l* enables one to calculate the acceleration due to gravity; for, from Eq. (7)

$$g = 4\pi^2 l/T^2 \qquad (8)$$

It is of interest to note that neither the amplitude of vibration nor the mass (or dimensions) of the ball appears in this equation. This means that the ball may be of any material and of any shape or size, subject only to the requirement that its diameter be small relative to the length of the string. Moreover, the period of vibration is the same for all amplitudes providing the amplitude is so small that the ratio *x/l* is at all times a good approximation to $\sin\theta$. For example, an amplitude of vibration of 5° requires significantly no more time for a complete vibration than a much smaller amplitude. (It is left for a problem at the end of the experiment to determine the range over which the approximation made in this derivation may reasonably be expected to hold.) It should be noted that the effect of air friction was not considered in the above derivation. Hence Eq. (7) is strictly true only for a simple pendulum that is vibrating in a vacuum.

It is of interest to consider the energy changes that occur during the course of any simple harmonic motion. When the vibrating particle passes through the midpoint of its motion it has its maximum kinetic energy, since its velocity is a maximum at that point. As the velocity decreases the kinetic energy is converted into potential energy. The sum of the kinetic energy and the potential energy remains constant throughout the vibration, assuming no losses due to friction or other causes. At the end of the path, where the velocity is zero, the potential energy is maximum; but at the midpoint of the vibration the potential energy is equal to the kinetic energy. The nature of the potential energy depends on the nature of the forces acting on the vibrating particle. For example, if the variation is that of a tuning fork, the potential energy is that due to forces of elastic deformation. In the case of a pendulum, the potential energy is due to the force of gravity. If the ball of a simple pendulum rises to height h above its lowest position its potential energy is *mgh* relative to the lowest point. This, then, is equal to the kinetic energy $\frac{1}{2}mv^2$ at the lowest point where v is the velocity at that point-neglecting, of course, small losses due to friction.

The simple pendulum is a special case of the more general *compound pendulum*, the name given to an object of any shape or size that is supported so as to be free to vibrate about a horizontal axis in the earth's gravitational field. The analysis of its motion becomes somewhat more complex than that for a simple pendulum. It turns out that the period of a compound pendulum is given by the equation

$$T = 2\pi\sqrt{I/(mgh)} \qquad (9)$$

where m is the mass of the object, I is its moment of inertia about the point of support and h is the distance between the point of support and the center of mass. When this equation is applied to a sphere of mass m and radius r supported as a simple pendulum of length l, I is equal to the moment of inertia of the sphere about its own axis ($2/5mr^2$) plus ml^2. When this is substituted in Eq. (9), the period is given by the equation

$$T = 2\pi\sqrt{\frac{l}{g}\left(1+\frac{2h^2}{5l^2}\right)} \qquad (10)$$

Note that this equation reduces to Eq. (7) if r is small relative to l. (Just how large r can be relative to l before it appreciably affects the answer is left as a problem to be solved at the end of the experiment.)

In this experiment it is desirable to determine the accuracy of the measurements and the calculations that are made. This can be done by applying their methods of handling errors described in Selective Experiment No. 001 ("Errors"). The following illustrates the application of these methods in this experiment.

To obtain a good value of the period T for a particular length of pendulum and amplitude of vibration the time of a fairly large number of vibrations is measured. This is repeated several times and the average is obtained. An estimate of the uncertainty in this result can be obtained as follows. In Table I, five measurements of the time for 25 vibrations of a particular pendulum are recorded. The average value by which each measurement differs from the mean, regardless of sign, is calculated. This is called the *average deviation* (the "a.d."). It is a reasonable measure of the uncertainty in the individual readings. The uncertainty in the average value of the time is, of course, less than this. Statistical theory says that it is less by the factor one to the square root of n, where n is the number of readings made; that is, by the factor $1/\sqrt{n}$. This is called the *Average Deviation of the Mean* (the " A.D."). The average value of the time can then be recorded together with its uncertainty, as shown in the Table. (The notation t = 15.42 ± 0.03sec simply means that the probability is that the correct time is between 15.39sec and 15.45sec. Statistical theory enables one to calculate the value of this probability if a large number of readings is taken) The value of the time for one vibration and also its uncertainty can then be calculated, and it is also recorded as shown. Note that the approximations are such that it is meaningless to retain more than one significant figure in the value of the uncertainty.

TABLE I

A method of recording a series of measurements for the convenience of calculating their average deviation from the mean. The accompanying calculations show how this is used to arrive at the approximate uncertainty in the measurement that was desired.

Trial No.	Time for 25 vib. t	Deviation from the mean
1	15.4 sec	0.02 sec
2	15.5	0.08
3	15.3	0.12
4	15.5	0.08
5	15.4	0.02
total	77.1	0.32
average	15.42	0.06

Calculations for data of Table I a.d. =0.06

$$\text{A.D.} = \frac{0.06}{\sqrt{5}}$$

= 0.03 sec

Therefore, time for 25 vib.

$$t = 15.42 \pm 0.03\,\text{sec}$$

Therefore, the period

$$T = 0.616 \pm 0.001\,\text{sec}$$

Finally, when numbers that have uncertainties assigned to them are manipulated arithmetically, the following rules for the "propagation of errors" may be applied:

1. In addition and subtraction, the uncertainty in the answer is the sum of the numerical uncertainties of the individual numbers.
2. In multiplication and division, the uncertainty in the answer is the sum of the percentage uncertainties of the individual numbers.

APPARATUS: A metal ball suspended by a length of string from a rigid support; a stopwatch or electric timer; two meter sticks and/or double length meter stick; protractor.

PROCEDURE: Suspend a metal ball from a rigid support by a light string using a pendulum support as shown in Fig. 3. The support must be sufficiently rigid so that no significant movement is imparted to it by the movement of the pendulum. If the support is not rigid the ball will not continue to vibrate in the vertical plane in which it is started, but will take on an elliptical path so that it becomes impossible to make an accurate determination of the period. For greatest rigidity the rod to which the pendulum support is clamped should be as large in diameter as possible and it should be securely attached to the laboratory table by means of a table clamp. Furthermore, the pendulum should be supported by the knurled thumbscrew and clamp that is closest to the rod rather than by either of those that are farther away.

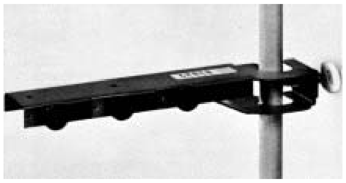

Fig. 3. Suitable clamp for supporting a simple pendulum.

Part I. **Variation of Period with Amplitude**. Adjust the length of the pendulum to about 60cm and measure its length (for use in Part II). Recall that the length of a simple pendulum is the distance from the point of support to the center of the ball. A quite accurate determination of this distance can be obtained by the use of a meter stick when the pendulum is hanging freely. Place the zero end of the meter stick against the point of support and simply read the meter stick where it touches the side of the ball when it is tangent to it. This distance is, of course, the hypotenuse of the right-angle triangle of which you really want to know the length of the long leg (from point of support to center of ball). However, because the angle between the meter stick and the string is so small the error is negligible. (Be sure to see that the zero end of the meter stick has not been worn off. If it has, take this into consideration in your determination of length.) Make an estimate of the uncertainty in your measurement. For example, if you read 61.3cm and estimate the uncertainty to be about 2mm, record the length as 61.3 ± 0.2cm.

Make an accurate determination of the period of the pendulum at a number of different amplitudes, for example the angular amplitudes (the maximum values of θ in Fig. 2) of about 1°, 2°, 4˚, 10° and 30°. In each case make four or five measurements of the time required for 25 vibrations, from which you then calculate the period T. Note that for the most accurate measurement of time you should both start and stop your timer when the pendulum passes through the midpoint of its vibration (moving in the same direction, of course), not when it is at one of the endpoints. (Explain why.) A reference line placed at the midpoint will assist you in anticipating this instant. Calculate the uncertainty in each value of T by using the method outlined above. Assemble your results in a table and explain how they demonstrate the theoretical analysis given above.

Part II. Variation of Period with Length. In the same manner measure the period of your pendulum for several different lengths- say at about 120cm, 80cm, 40cm and 20cm and at a small angular amplitude of no more than about 5° in each case. Note that it is the angular amplitude of the pendulum and not simply the linear amplitude that must be kept small. (The approximation that sin θ be equal to *x/1* does not hold throughout the vibration if the angular amplitude is too large.) Calculate the uncertainty in your measurement of T for each length and assemble your results in a table as suggested by Table II. (Record in the table the uncertainty along with each result.) Calculate the values of T^2 *and* l/T^2 for each length and calculate their uncertainties by using the rules for the propagation of errors given above. (Note that Rule 2 applies to the calculation of the uncertainty in the value of T^2 as well as in the value of l/T^2.)

TABLE II

A series of measurements of the period T of a simple pendulum as a function of its length l.

Length l	Period T	T^2	l/T^2

Part III. Calculation of the Acceleration due to Gravity. Calculate the average value of l/T^2 from Table II and from this calculate the acceleration due to gravity g. Calculate the uncertainty in the value of g by first calculating the percent uncertainty and then the numerical uncertainty. Record both values of the uncertainty. Finally, compare your result with the accepted value for the acceleration due to gravity at your location.

It is desirable to display the results of your experiment graphically. Plot two curves on the same sheet of paper:

(1) period T as a function of length l as abscissa and

(2) T^2 as a function of length. What do these graphs demonstrate?

PROBLEMS: 1. Compute the ratio of x to l in Fig. 2 for an angle of 5° and also for an angle of 30°. Explain how your results are related to (confirm?) the assumptions made in the derivation of Eq. (7).

2. A so-called "seconds pendulum" is one that passes through its equilibrium position once each second. Calculate the length of such a simple pendulum.

3. Take the derivatives of the expression for x given by Eq. (4) with respect to time twice and compare your result with Eq. (3). Show that $\omega^2 = k/m$.

4. The derivation given above for the period of a simple pendulum is strictly true under vacuum conditions. Discuss the effect of air friction on the motion of a simple pendulum. Explain the effect on the period.

5. If the maximum elevation to which the ball of a simple pendulum rises is h, show that its velocity at the midpoint in its vibration is the same as if it had fallen freely vertically through the distance h; that is, show that $v = \sqrt{2gh}$.