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THE COMPOUND PENDULUM

OBJECT: To study the properties of a compound pendulum, and to determine the acceleration due to gravity by the use of such a pendulum.

METHOD: An experimental pendulum is suspended successively about several axes at different points along its length and the period about each axis is observed. A graph is plotted of the period versus the distance of the axis of suspension from one end of the pendulum. The nature of the graph shows the physical properties of the compound pendulum. From values of the period and the corresponding length of the equivalent simple pendulum as determined from the graph, the acceleration due to gravity is calculated. From the mass of the pendulum and its radius of gyration as obtained from the curve, the rotational inertia of the pendulum is computed.

THEORY: A *simple* pendulum consists of a small body called a "bob" (usually a sphere) attached to the end of a string the length of which is great compared with the dimensions of the bob and the mass of which is negligible in comparison with that of the bob. Under these conditions the mass of the bob may be regarded as concentrated at its center of gravity, and the length *l* of the pendulum is the distance of this point from the axis of suspension. When a simple pendulum swings through a small arc, it executes linear simple harmonic motion of period *T*, given by the equation

$$T = 2\pi\sqrt{\frac{l}{g}} \tag{1}$$

where *g* is the acceleration due to gravity. This relation-ship affords one of the simplest and most satisfactory methods of determining *g* experimentally.

When the dimensions of the suspended body are not negligible in comparison with the distance from the axis of suspension to the center of gravity, the pendulum is called a *compound*, or *physical*, pendulum. Any body mounted upon a horizontal axis so as to vibrate under the force of gravity is a compound pendulum. The motion of such a body is an angular vibration about the axis of suspension. The expression for the period of a compound pendulum may be deduced from the general expression for the period of any angular simple harmonic motion

$$T = 2\pi\sqrt{\frac{-\theta}{a}} \tag{2}$$

and the application of Newton's second law of motion for rotating bodies

$$L = Ia$$

where θ is the angular displacement, α the angular acceleration, *L* the torque and *I* the rotational inertia of the body.

Figure 1 represents a compound pendulum of mass m, consisting of a rectangular bar AB to which a cylindrical

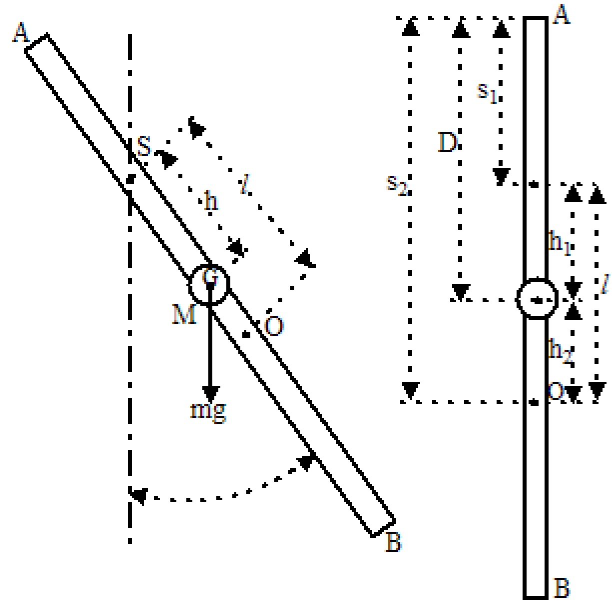

Fig. 1. Compound pendulum

mass M is attached. The pendulum is suspended on a transverse axis through the point S. In the diagram, the cylindrical mass M is represented as being exactly in the middle of the bar, thereby making a symmetrical system in which the center of gravity G is at the geometrical center. Obviously, this particular condition is a very special case, and has nothing whatever to do with the general treatment of the problem.

In the equilibrium position, the center of gravity G is vertically below the axis of suspension S. When the body is rotated through an angle θ, the weight of the system *mg*, which may be regarded as concentrated at the center of gravity, exerts a restoring torque about S given by

$$L = mgh\sin\theta \tag{4}$$

where h is the distance from the axis of suspension to the center of gravity. If a minus sign is used to indicate the tact that the torque L is opposite in sign to the displacement θ, Eqs. (3) and (4) yield

$$Ia = -mgh\sin\theta \tag{5}$$

When the angular displacement θ is sufficiently small, sinθ is approximately equal to θ measured in radians. With this restriction Eq. (5) may be written

$$a = \frac{-mgh}{I}\theta \tag{6}$$

Since m, g, h and I are all numerically constant for any given case, Eq. (6), may be written simply

$$a = -k\theta \tag{7}$$

where k is a constant. Equation (7) is the defining equation of angular simple harmonic motion, i.e., motion in which the angular acceleration is directly proportional to the angular displacement and oppositely directed. Since the system executes angular simple harmonic motion, substitution of the expression for a from Eq. (6) in Eq. (2) yields the equation for the period of a compound pendulum

$$T = 2\pi\sqrt{\frac{I}{mgh}} \tag{8}$$

where I is the rotational inertia of the pendulum about the axis of suspension S. It is convenient to express I in terms of I_0, the rotational inertia of the body about an axis through its center of gravity G. If the mass of the body is m,

$$I_o = mk_o^{\,2} \tag{9}$$

where k_o is the radius of gyration about an axis through G. For any regular body, k_o can be computed by means of the appropriate formula (see any handbook of physics or engineering); for an irregular body it must be determined experimentally. The rotational inertia about any axis parallel to the one through the center of gravity is given by

$$I = I_o + mh^2 \tag{10}$$

where h is the distance between the two axes. Substitution of the relationships of Eqs. (9) and (10) in Eq. (8) yields

$$T = 2\pi\sqrt{\frac{k_o^{\,2} + h^2}{gh}} \tag{11}$$

This equation expresses the period in terms of the geometry of the body. It shows that the period is independent of the mass, depending only upon the distribution of the mass (as measured by k_o) and upon the location of the axis of suspension (as specified by h). Since the radius of gyration of any given body is a constant, the period of any given pendulum is a function of h only. Comparison of Eq. (1) and Eq. (11) shows that the period of a compound pendulum suspended on an axis at a distance h from its center of gravity is equal to the period of a simple pendulum having a length given by

$$l = \frac{k_o^{\,2} + h^2}{h} = h + \frac{k_o^{\,2}}{h} \tag{12}$$

The simple pendulum whose period is the same as that of a given compound pendulum is called the "equivalent simple pendulum."

It is sometimes convenient to specify the location of the axis of suspension S by its distance s from one end of the bar, instead of by its distance h from the center of gravity G. If the distances s_1, s_2 and D (Fig. 1) are measured from the end A, the distance h_1 must be considered negative, since h is measured from G. Thus, if D is the fixed distance from A to G, $s_1 = D + h_1$, $s_2 = D + h_2$ and, in general, $s = D + h$. Substitution of this relationship in Eq. (11) yields

$$T = 2\pi\sqrt{\frac{k_o^{\,2} + (s-D)^2}{g(s-D)}} \tag{13}$$

The relationships between T and s expressed by Eq. (l3) can best be shown graphically. When T is plotted as a function of

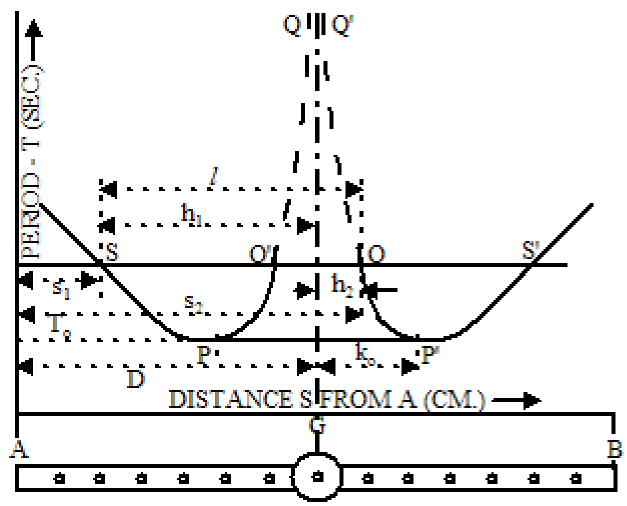

Fig. 2. Graphical analysis of the compound pendulum

s, a pair of identical curves SPQ and S'P'Q' are obtained as illustrated in Fig. 2. (The dotted portions represent extrapolations over apart of the body where it is difficult to obtain experimental data with this particular pendulum.) Analysis of these curves reveals several interesting and remarkable properties of the compound pendulum. Beginning at the end A, as the axis is displaced from A toward B the period diminishes, reaching a minimum value at P, after which it increases as S approaches G. The two curves are asymptotic to a perpendicular line through G, indicating that the period becomes infinitely great for an axis through the center of gravity. As the axis is displaced still farther from A (to the other side of G), the period again diminishes to the same minimum value at a second point P', after which it again increases.

A horizontal line SS', corresponding to a chosen value of *T*, intersects the, graph in four points, indicating that there are four positions of the axis, two on each side of the center of gravity, for which the periods are the same. These positions are symmetrically located with respect to G. There are, therefore, two numerical values of *h* for which the period is the same, represented by h_1 and h_2 (Figs. 1 and 2). Thus, for any chosen axis of suspension S there is a conjugate point O on the opposite side of the center of gravity such that the periods about parallel axes through S and O are equal. The point O is called the *center of oscillation* with respect to the particular axis of suspension S. Consequently, if the center of oscillation of any compound pendulum is located, the pendulum may be inverted and supported at O without altering its period. This so-called reversibility is one of the unique properties of the compound pendulum and one that has been made the basis of a very precise method of measuring *g* (Kater's reversible pendulum).

It can be shown that the distance between S and O is equal to *l*, the length of the equivalent simple pendulum. Equating the expressions for the squares of the periods about S and O, respectively, from Eq. (11)

$$T^2 = \frac{4\pi^2}{g}\left(\frac{k_o^{\,2} + h_1^{\,2}}{h_1}\right) = \frac{4\pi^2}{g}\left(\frac{k_o^{\,2} + h_2^{\,2}}{h_2}\right) \qquad (14)$$

from which

$$T^2 = \frac{4\pi^2}{g}\left(h_1 + h_2\right) \qquad (15)$$

or

$$T = 2\pi\sqrt{\frac{h_1 + h_2}{g}} \qquad (16)$$

Comparison of Eqs. (1) and (16) shows that the length *l* of the equivalent simple pendulum is

$$l = h_1 + h_2 \qquad (17)$$

Thus, the length of the equivalent simple pendulum is SO (Figs. 1 and 2).

S' and O' are a second pair of conjugate points symmetrically located with respect to S and O respectively, i.e., having the same numerical values of h_1 and h_2.

Further consideration of Fig. 2 reveals the fact that the period of vibration of a given body cannot be less than a certain minimum value T_o, for which the four points of equal period reduce to two, S and O' merging into P, and S' and O merging into P' as h_1 becomes numerically equal to h_2. The value of h_o corresponding to minimum period can be deduced by solving Eq. (14) for k_0^2, which yields

$$k_o^{\,2} = h_1 h_2 \qquad (18)$$

and setting

$$h_o = h_1 = h_2 \qquad (19)$$

Thus

$$h_o = k_o \qquad (20)$$

Substituting in Eq. (12) yields

$$l_o = 2k_o \qquad (21)$$

Thus, the shortest simple pendulum to which the compound pendulum can be made equivalent has a length l_o equal to twice the radius of gyration of the body about a parallel axis through the center of gravity. This is indicated in Fig. 2 by the line PP'. Inspection of the figure further shows that, of the two values of *h* for other than minimum period, one is less than and one greater than k_o. From the foregoing it is evident that if two asymmetrical points S and O can be found such that the periods of vibration are identical, the length of the equivalent simple pendulum is the distance between the two points, and the necessity for locating the center of gravity is eliminated. Thus, by making use of the reversible property of the compound pendulum, a simplicity is, achieved similar to that of the simple pendulum, the experimental determinations being reduced to one measurement of length and one of period.

APPARATUS: The apparatus used in this experiment is very simple, consisting merely of a rectangular steel bar approximately 1 meter long carrying a heavy cylindrical mass, and supported on a horizontal axis (Fig. 3). The bar

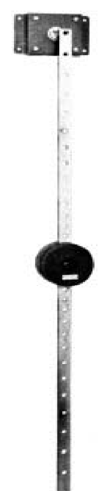

Fig. 3. Experimental Pendulum

has a series of holes distributed along its length to provide several axes of suspension. In use the pendulum is supported successively at the various holes on a hardened steel knife-edge in a wall bracket, and its period of vibration determined with the aid of a stopwatch. A meter stick and a platform balance with a set of weights are the only other apparatus required.

PROCEDURE:

Experimental: Support the pendulum on the knife-edge at the hole nearest to one end of the bar, making sure that it swings freely in a vertical plane. With the aid of a stopwatch, observe the time of 50 full vibrations and determine the period. In making this determination, begin with the count of "zero" as the pendulum passes through its central position, count "one" as it makes its next transit through center *going in the same direction*, etc. In a like manner determine the period about an axis through each of the several holes.

Remove the pendulum from its support and with a meter stick (preferably one equipped with caliper jaws) measure the distances s_1, s_2, etc., of the various points of suspension from one end of the bar. Record these lengths opposite the corresponding periods.

Weigh the pendulum on the platform balance and record its mass m.

Analysis of Data: 1. Plot the data in a graph similar to Fig. 2. Draw any horizontal line SS'. From the corresponding period T as determined by the ordinate of this line, and the length l of the corresponding equivalent simple pendulum as given by the average of the values of SO and S'O', calculate the acceleration g due to gravity, by means of Eq. (1) . Compare with the accepted value and record the percentage difference.

2. From the mass m of the pendulum and the radius of gyration k_o as determined from the graph, compute the rotational inertia I_o about the axis G by Eq. (9). Compute the rotational inertia I about the axis S by Eq. (10).

QUESTIONS: 1. What is the minimum period with which this pendulum can vibrate? What is the length of a simple pendulum having the same period?

2. Describe how Fig. 2 would be altered if the cylindrical mass M were near one end, say the end B.

3. With a given, axis of suspension, say S, discuss the effect upon the period of (a) increasing the mass of the cylindrical body; (b) moving it nearer to S.

4. How would the value of the minimum period T_o be affected by moving the mass M in either direction from the middle?

5. With the mass M near the end B and the pendulum suspended about an axis S near A, how could the vibration of the system about the axis S' be experimentally observed?

6. Does the center of oscillation of a solid body, such as a rod or bar, lie within the body for any transverse axis of suspension? Explain.

7. Locate the center of oscillation of a meter stick suspended about a transverse axis at the 10cm mark. At what other positions could the meter stick be suspended and have the same period?

8. Prove that the period of a thin ring hanging on a peg is the same as that of a simple pendulum whose length is to the diameter of the ring.