• C.J. Overbeck Northwestern University • R.R. Palmer Beloit College • Marsh W. White The Pennsylvania State University

SIGNIFICANT FIGURES; PROPAGATION OF ERRORS

OBJECT: To study the errors and their propagation from data taken in a simple experiment.

METHOD: Some elementary measurements are made with simple apparatus, and the errors in the data are estimated. The data and consequent calculations are examined with regard to the proper use of significant figures and the propagation of errors when the data are arithmetically manipulated. The uncertainties in the measurements are examined to determine the average deviation of a single observation and the average deviation of the mean of a number of observations. The distribution of random errors is illustrated by the careful weighing of a number of roughly-calibrated 100gm masses.

THEORY: Laboratory measurements usually involve the comparison of a numerical observation to a standard value. Since there are uncertainties or "errors" in all measurements the accuracy of such observations is limited by the degree of refinement of the measuring device and the skill of the observer. The experimenter wishes to know not only the magnitude of the given quantity but also the accuracy with which the measurement has been made. The technical terminology used in these procedures will be only briefly outlined here but may be studied more fully in the standard treatises on the treatment of errors*

*More extensive references to the treatment of errors can be found in the following:
Lyman G. Parratt, *Probability and Experimental Errors in Science*. (John Wiley and Sons, Inc., 1966)
E. M. Pugh and G. H. Winslow, *The Analysis of Physical Measurements*. (Addison Wesley Publishing Co., Inc., 1966)
Yardley Beers, *Introduction to the Theory of Errors*. (Addison Wesley Publishing Co., 1957)
Selective Experiments in Physics, Number 001. (Central Scientific Co.)

I. Significant Figures. Every digit in a number is significant unless it is used merely to locate the decimal point. In scientific work only figures that are reasonably trust-worthy are retained. In recording data it usually is customary to retain only one estimated or doubtful figure. Therefore, if a measurement is taken with accuracy estimated in the second decimal place, it should be recorded to that extent, as for example 20.00cm (four significant figures) and *not* 20cm. Zeros used merely to locate the decimal point are not regarded as significant figures. For example the measurement 2.35cm could be recorded as 0.0235m or 0.0000235km, each with three significant figures. (To avoid confusion it would be better to record the latter two numbers as 2.35×10^{-3} or 2.35×10^{-5}km.) When computations are made with observed data, figures in the result that are not significant should be discarded. For example if observed data such as 10.77 and 3.55 (with the last digit doubtful but considered significant) are multiplied the product 38.2335 should be recorded as 38.2. In adding and subtracting numerical data it is misleading to retain the figures in a column that has an unknown digit. The following rough rules may be used for arithmetical computations:

Rule A. In addition and subtraction, carry the result only through the first column that contains a doubtful figure.

Rule B. In multiplication and division, write the result as a number which, as nearly as possible, has the percent-age accuracy of the least accurate figure in the computation.

II. Errors. The uncertainty of an observed piece of data technically is referred to as the *error*. (This term "error" carries no implication of mistake or blunder; it means the uncertainty between the measured value and the standard value.) An error that tends to make a reading too high is called a positive error and one that makes it too lowa negative error.

Errors may be grouped into two classes, systematic and random. A *systematic error* is one that always produces an error of the same sign, for example, one that would tend to make all the observations too small. A *random error* is one in which positive and negative errors are equally probable.

Systematic errors may be subdivided into three groups: Instrumental, personal, and external. An *instrumental error* is caused by faulty or inaccurate apparatus, for example, an undetected zero error in a scale or an incorrectly adjusted watch. *Personal errors* are due to some bias of the observer, such as maintaining faulty position of the eye relative to a scale. *External errors* are caused by external conditions (wind, temperature, humidity, vibration); an example is the expansion of a scale as the temperature rises. Corrections are made for systematic errors when they are known to be present.

Random or erratic errors occur as variations that are due to a large number of factors, each of which adds its own contribution to the total error. Inasmuch as these factors are unknown and variable it is assumed that the resulting error is a matter of chance, and therefore positive and negative errors are" equally probable. For example, random deviations are observed in the patterns obtained when an archer is shooting at the center of a target. Because random errors are subject to the laws of chance their effect in the experiment may be lessened by taking a large number of observations.

III. Deviation of Values from their Mean. Since the variations involved in observations subject to random errors are governed by chance, one may apply laws of statistics to them and arrive at certain definite conclusions about the magnitude of the errors. These laws lead to the conclusion that the value having the highest probability of being correct is obtained by dividing the sum of the individual readings to the total number *n* of observations. This value is called the arithmetic mean, a.m.

The difference between an observation and the arithmetic mean is called the *deviation d*. The *average deviation* a.d. is a measure of the accuracy of the observations. The average deviation is the sum of the deviations *d* (without regard to sign) divided by the number *n* of observations, or $a.d. = \sum d / n$. It is known from the theory of probability that an arithmetic mean computed from n equally reliable observations is on the average more accurate than anyone observation by a factor of $\sqrt{n}$ to 1. Consequently, the Average Deviation A.D. of the mean of *n* observations is given by

$$A.D. = \frac{a.d.}{\sqrt{n}}$$

For example, in the measurement of a block of wood, suppose several trials give the following data

Length cm	Deviation cm
12.32	-0.03
12.35	0.00
12.34	-0.01
12.38	+0.03
12.32	-0.03
12.36	+0.01
12.34	-0.01
12.38	+0.03
Mean: 12.35	$\sum d = 0.15$

The best average value of this set of observations should be written 12.35 ± 0.01cm.

Percentage Error. The importance of an error in an experimental value lies not in its absolute value but in its relative, or percentage, value. By percentage error is meant the number of parts out of each 100 parts that a number is in error:

$$\text{Percentage error} = \frac{error}{s\tan dardvalue} \times 100\%$$

Percentage errors are usually wanted to only one or two significant figures, so that the method of mental approximation or a rough slide-rule computation is quite sufficient for most purposes.

It frequently happens that the percentage difference between two quantities is desired when neither of the quantities may be taken as a "standard value." In such cases their average value may be used in place of a standard value.

IV. Propagation of Errors. Since the uncertainties in various numerical observations introduce errors in the final result, it is essential to utilize certain rules that have been developed in the theory of the propagation of errors. Although the more exact theory of probable errors will not be used in this experiment, the following approximate rules may be applied to computations involving errors in observed data:

Rule C. The numerical error of a sum or difference is the sum of the numerical errors of the individual quantities.

Rule D. The percentage error of the product or quotient of several numbers is the sum of the percentage errors of the several quantities entering into the calculation.

Certain numbers that commonly appear in calculations have a peculiar relationship in that they appear by definition rather than by measurement. Such numbers are assumed to have an unlimited number of significant figures. The numbers 2 and π in the expression 2πr for the circumference of a circle are examples of such numbers; also many conversion factors, such as 60 sec/min.

APPARATUS: Triple-beam balance; trip-scale balance; accurate meter stick; good quality millimeter ruler; ten rough 100gm masses (slotted or hook "weights"); special wooden block, about 25 by 10 by 5cm.

PROCEDURE: 1. Determine the thickness of a page in one of your textbooks by first placing apiece of colored paper between pages near the beginning and another piece near the end of the book and then measuring their separation with a good millimeter ruler. Make one measurement with 500 hundred pages and another with 100 pages. Calculate the thickness of one page for each observation. How many significant figures do you have in your result? How do the two values of the thickness of one page compare with regard to their accuracy?

2. Examine a sensitive triple-beam balance that is accurate to the nearest centigram. The balance has a hardened-steel knife-edge that may be lowered by a knurled-screw beam arrest onto an agate bearing. The arrest should always be raised when heavy masses are added to the pan. Check the zero position of the balance. Adjust the balance screw so that, with no load in the pan and the riders all set at zero, the beam swings equally on either side of the zero of the scale. In weighing it is not necessary to allow the beam to come to rest, but the masses should be adjusted to make the oscillations symmetric about the zero. Select 10 or more rough iron masses marked 100 gm, hut which are actually only approximately that value. Measure the mass of each of the "weights". Record the positive or negative numerical error between the observed mass and the value stamped on it, assuming the balance readings to be correct. (Is this assumption justified?) Are these errors random or systematic? If the error is systematic try to explain its origin. Calculate the a.d. in the measurement of each mass. Determine the A.D. of the mean of these measurements. Express this as a percentage error.

3. The density (mass/volume) of a wooden block should next be determined, from observations of its length, breadth, thickness, and mass. A special block should be used, cut in such a manner that none of its dimensions is precisely "true" so that the observations will have exaggerated errors in order to facilitate the ideas of this exercise.

(a) Measure and record the length of the block, using the millimeter scale. Take nine observations at different places on the block in order that a fair average may be obtained. In each reading estimate fractional parts of millimeter divisions. (Remember to include the proper number of zeros when the observed length seems to fall exactly on a scale division.) The best technique for making careful measurements with a rule involves placing the rule with the graduations immediately adjacent to the object to be measured, as in Fig. 1, and arranging the scale to coincide with one edge of the block at the 1cm or the 10cm mark. This arrangement minimizes some of the error due to parallax and to the worn condition of the zero end of most rules.

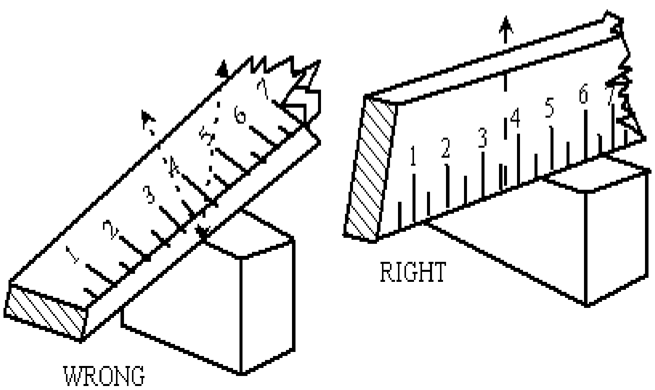

Fig. 1. Precautions in the use of rulers.

Calculate the mean of the observations of the length. Find the deviations of the observed lengths from the mean and calculate the a.d. Compute the A.D. of the mean length. Record this mean and its A.D. as the observed length of the block.

(b) Repeat Step (a) for the breadth and thickness of the block and determine the A.D. of each.

(c) Calculate the volume of the block, recording the result to the proper number of significant figures. Determine the percentage error and then the numerical error of this volume, using for the respective errors of the length, breadth, and thickness the A.D.'s found above.

(d) Determine the mass of the block by means of the trip-scale balance. Estimate the uncertainty in the determination by noting the smallest change of mass that will produce a readable change in the balance. Compute the percentage of this uncertainty. Compare this percentage uncertainty of the mass with the percentage uncertainty of the volume. Is it negligible? If so, was it worthwhile to weigh the block as accurately as you did? How can preliminary estimates of the uncertainty of one factor be used to determine the apparatus and procedure that should be used in measuring another quantity?

QUESTIONS: 1. In Step 3 (d) of this experiment what other errors might have been present? Would you expect these to be larger than the error observed because of the sensitivity of the balance?

2. Calculate the percentage difference between one cubic centimeter and one milliliter if one liter is equal to 1.000028 cubic centimeters.

3. The standard kilogram is equal to 2.2046 pounds mass. What percentage error is made if this number is rounded off to be 2.2?

4. Classify the following as to whether they are systematic or erratic errors: (a) incorrect calibration of scale; (b) personal equation or prejudice; (c) expansion of scale due to temperature changes; (d) estimation of fractional parts of scale divisions; (e) displaced zero of scale; (f) pointer friction; (g) lack of exact uniformity in object repeatedly measured; (h) parallax.

5. The masses of three bodies, together with their respective errors, were recorded as follows: m_1 = 31.47.226 ± 0.3*gm*; m_2 = 8.3246*gm* ± 0.10%; m_3 = 604.279*gm*, error 2 parts in 5000. Assuming that the errors are correctly given, (a) indicate any superfluous figures in the measurements; (b) compare the precision of the three quantities; (c) find their sum; (d) properly record their product.

6. Assuming that the following numbers are written with the correct number of significant figures, make the indicated computations, carrying the answers to the correct number of significant figures: (a) add 372.6587, 25.61, and 0.43798; (b) 24.01 x 11.3 x 3.1416; (c) 3887.6 x 3.416/25.4

7. The measured dimensions of a rectangular block are 2.267 ± 0.002in., 3.376 ± 0.002in., and 0.207 ± 0.001in. Compute the volume of the block and record the result with the correct number of significant figures.

8. If you were new at a measuring job and time was pressing what might be the minimum number of observations you would try to make? Explain. If you had made four observations how would you decide whether to stop or to make more?

9. In a determination of the mass of a metal disk a balance and a set of masses guaranteed to an accuracy of ± 2.0gm were used. Six different observers recorded the following values:

A. 4,044gm; B. 4040gm; C. 4042gm; D. 4024gm; E. 4,043gm. What is the mean, the a.d., and the A.D. of these values?

Do you have reason to suspect that observer D made a mistake in his observation? Omit this observation and re-calculate the above. Which of these results do you prefer? Why?