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GRAPHS AND EQUATIONS

INTRODUCTION: Experimental work in science is frequently a study of the relationship between two interacting variables. For example the following questions might be answered from experimental data. How does the velocity of a falling body vary with time? What is the angular distribution of radiant energy transmitted through a small opening? What is the pressure response frequency characteristic of a crystal telephone receiver?

When experiments are performed, the *independent variable*, in these examples time, angle, and frequency, is progressively changed, and the corresponding values of the resulting *dependent variable*, velocity, intensity, and response respectively, are measured for a series of tests. These data are appropriately recorded in an organized table, that is, in tabular form.

A display of the data as a graph shows more clearly than the tabular form how the one quantity, or property, is related to the second. The graph also indicates probable experimental errors and provides values intermediate to the several readings.

The most powerful form in which the relationship of the variables can be expressed is a mathematical equation. Such equations permit various mathematical expansions and the deduction of additional information. A straight line curve on a graph may be converted quickly to the equation form. Obtaining a straight line curve may require the selection of suitable conversion factors for the axis values, or a special type of graph paper. These techniques, the basis of this discussion, reduce the laborious matching of curves of empirical equations to a reasonable "match" of the original graph form.

GRAPHICAL PRESENTATION OF DATA: Various types of graph paper are available for the presentation of data. Each type has its own specific advantages. The three types of graph paper most commonly used are *rectangular coordinate* paper, *polar coordinate* paper, and *logarithm* paper. The latter is usually called log paper. These papers will be used to illustrate the three examples or questions mentioned in the Introduction.

Typical laboratory data for the first problem, "How does the velocity of a falling body vary with time?", are given in tabular form in Table I. At time t = 0 the initial velocity reading v_o was 15.0m/sec.

The graph (Fig. 1) of these data on *rectangular coordinate paper* shows a steady increase in velocity during the 5 sec of free fall. The velocity at 2.6 sec may be read from the graph as 40.5m/sec, and the expected velocity at 5.5 sec is 69.0 m/sec. Evidently the readings taken at 2 sec and 4 sec have the greatest experimental error. They do not fall on the straight line indicated by the other readings.

Table I.

VELOCITY IN FREE FALL.

t (sec)	v (m/sec)
0	$15.0 = v_o$
1	24.8
2	34.1
3	44.4
4	55.0
5	64.0

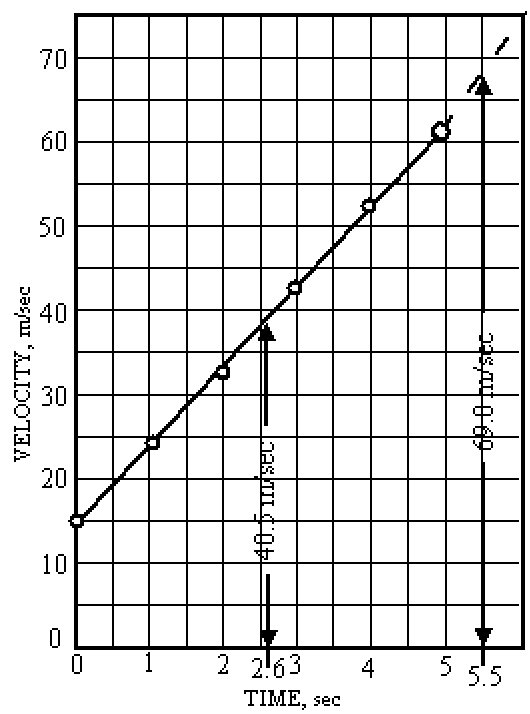

Fig. 1. Variation of velocity with time for free fall.

The angular distribution of radiant energy, sound or light, transmitted through a small opening is shown best on polar coordinate paper. See Fig. 2. The plane wave energy of wavelength λ approaches the opening of width *w* from the bottom. The relative intensity of the radiation propagated at each angle is plotted. Thus, for Fig. 2(a), where the wavelength equals the width of the opening, that is $\lambda = w$, the intensity at 20° from the normal is approximately 70% of the intensity measured at 0°. The intensity value has fallen to about half its maximum value at an angle of 35°.

Note that the major portion of wave energy transmitted through this small opening is in the forward direction. Figure 2(b) shows that when the size of the opening relative to the wavelength becomes larger, the energy flow not only concentrates into a narrower forward cone, but also produces secondary minima and maxima on either side of the opening. Notice the large difference in the two graphs when the angle is 30°. For $\lambda = w$ the intensity is about 60% of maximum, whereas the intensity at the same angle for

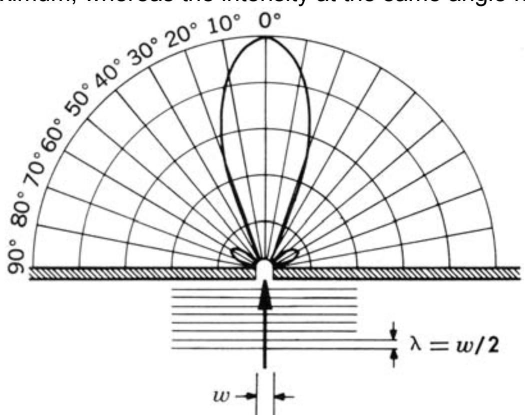

Fig. 2 (b). $w = 2\lambda$

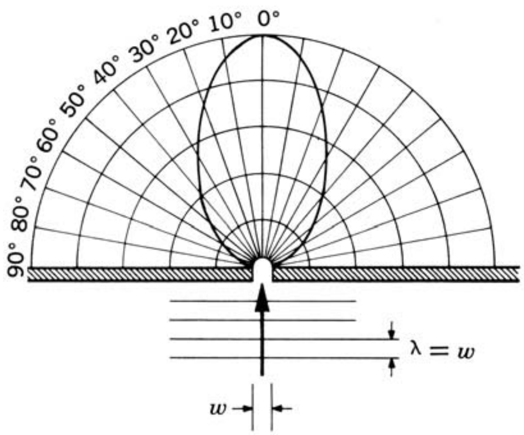

Fig. 2. (a). $w = \lambda$

$\lambda = w/2$ is almost zero. This intensity distribution is the result of diffraction and interference. The field of distribution is made very evident when the data are presented in graphical form.

Polar coordinate paper of 360° is selected to show the angular distribution of radiant power from a source as, for example, the luminous flux from a light source or the power in a sound wave radiated by aloud speaker. The essential feature in the use of polar coordinate paper for such purposes is the clearness of its visual presentation.

Data for the third problem stated in the Introduction, "What is the pressure response frequency characteristic of a crystal telephone receiver?", are plotted in Fig. 3. Since the hearing response of the ear is nearly proportional to the logarithm of the physical intensity of the sound, the ordinate is plotted in decibels. To accommodate the great frequency range of the ear and the relative contribution of these frequencies to our hearing, the abscissa points are plotted here on a logarithmic scale of three cycles, 10^1 to 10^2, 10^2 to 10^3 and 10^3 to 10^4. For this scale the linear distances are proportional to the logarithms of the graduated coordinate scale. This method of plotting data on semi-log paper gives a visual picture comparable to the sounds heard by the ear.

Note that this receiver has a low response for frequencies under 250vib/sec. Its highest response is in the range 300 to about 6000vib/sec, with relatively large fluctuations. Since intelligibility of speech lies in this range, this receiver would be acceptable for telephone conversation but certainly no "Hi-Fi" enthusiast would select it as a component of his equipment.

If the response had been measured in watts per square centimeter instead of decibels, a comparable graph to Fig. 3 could have been plotted directly on log-log paper, which has a logarithm scale for both the *x* and the *y* axes. The plot on log-log paper would give similarly the most meaningful picture of the response of the receiver relative to hearing.

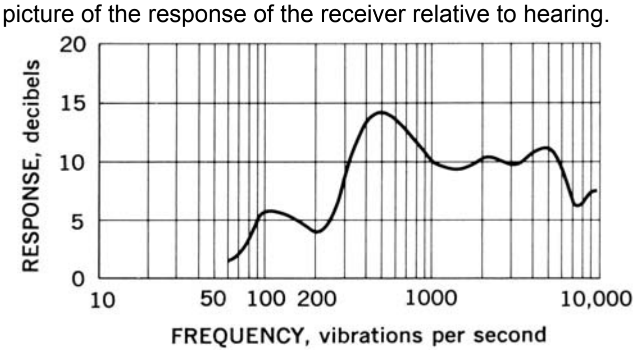

Fig. 3. Pressure response-frequency characteristic of a receiver.

EMPIRICAL EQUATIONS: The most powerful form of expressing an observed experimental relationship is an equation. This is a relatively easy task when the plotted curve is a straight line. If the curve is not a straight line the task becomes more difficult and may even require a laborious series of attempts to "match" test equations to the curve. The experimenter would not spend time trying to match an equation to the curve of Fig. 3 for several reasons: (a) each receiver has its own particular curve; so little is gained by even a successful attempt; (b) the curve alone gives an adequate picture of the response; (c) the unique form of the curve shows that it is practically an impossible job.

LINEAR RELATIONSHIP: Figure 4 is another plot of the data of Table I. The straight line shows a linear relationship between velocity and time. The equation relating these variables is, therefore, of the form $y = a\ x + b$. In this equation ***a*** is the *slope* of the line and ***b*** is its *y intercept*.

The slope *dv*/*dt* of the line of Fig. 4 is calculated as follows.

$$\frac{dv}{dt} = cons\tan t = \frac{(64.0 - 15.0)m/\sec}{(5-0)\sec} = 9.80 m/\sec^2$$

This number is recognized in free fall as the acceleration due to gravity for which the symbol *g* is used.

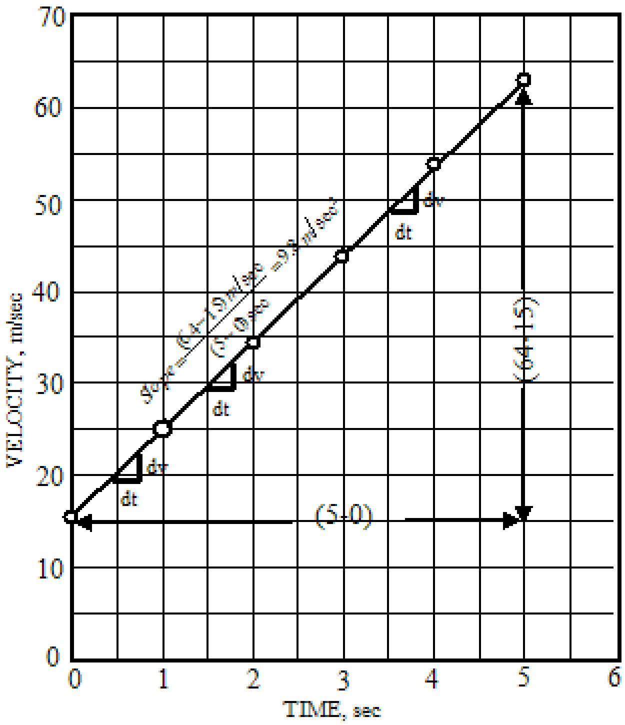

Fig. 4. Velocity-Time relation for constant acceleration.

Thus

$$\frac{dv}{dt} = g \quad (1)$$

$$dv = gdt$$

$$v = gdt$$

The value of the constant *b* is the velocity when *t* = 0. This initial velocity is designated v_o and is the velocity intercept of the curve [v_o = 15.0 m/sec].
Hence the equation of the straight-line curve of Fig. 4 is

$$v = v_o + gt = \frac{15m}{\sec} + \frac{9.80m}{\sec^2} \times t \quad (2)$$

Frequently when the initial equation is obtained, other information becomes available, through mathematical processes. Proceeding, we obtain

$$v = \frac{ds}{dt}$$

$$ds = vdt = (v_o + gt)dt$$

or

$$s = v_o t + \frac{1}{2} gt^2 + c \quad (3)$$

where *c* is the constant of integration and may be evaluated from boundary conditions of *t* = 0, *s* = 0, hence *c* = 0.
Thus, obtaining the equation of the curve leads also to a simple expression for the distance s of free fall at any time t for the falling body of the data given in Table I. At a time *t* = 5 sec, this body will have fallen a distance

$$s = (15 \times 5)m + \tfrac{1}{2}(9.8)m/\sec^2 (5)^2 \sec^2$$

$$s = 75m + 122.5m = 197.5m$$

Starting from rest, that is v_o = 0, the distance *s* in 5 sec would be 122.5m. The values in Table II were computed by the use of this equation.

NON-LINEAR RELATIONSHIP: The data for many relationships in science, when plotted on rectangular coordinate paper, do not produce a straight line curve of the type *y* = *a x* + *b*. An attempt is then made to find a method of plotting which will give a straight line. Experience and judgment are valuable elements in this operation of recognizing the original curves as hyperbolas, parabolas or some form of power or exponential curve. Each will be discussed in turn.

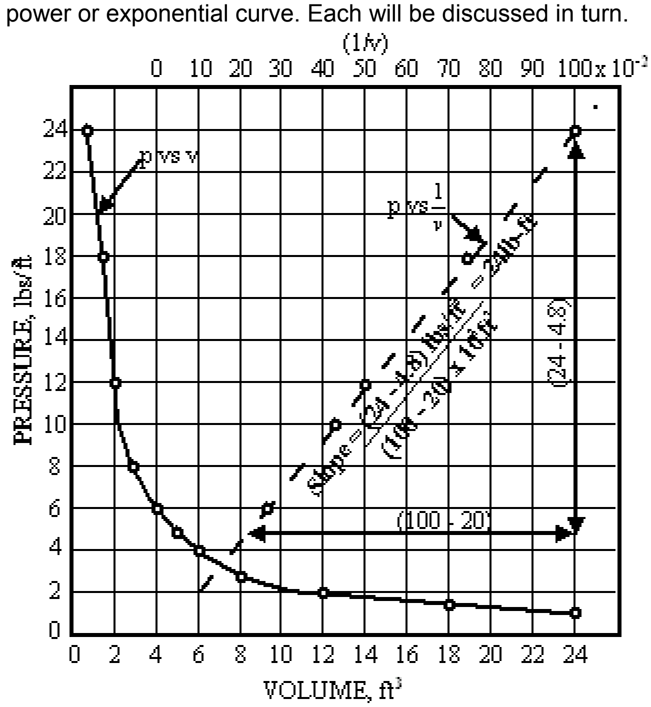

Fig. 5. The hyperbola, Boyle's law.

Rectangular hyperbola: The solid line curve of Fig. 5 represents the pressure-volume relationship of a given mass of gas at constant temperature. Noting the symmetry of this curve should suggest that it might be a rectangular hyperbola expressed by the equation

$$p\,v = c \quad (4)$$

This equation may be rewritten in linear form as *p* = *c* (l/*v*) where c is the slope of the straight line.
The test of this suspected form would be to plot pressure versus the reciprocal of the volume to see whether or not a straight line curve is obtained. This plot of *p* vs. l/*v* is the straight dash line of Fig. 5 having a slope of 24lb.-ft. Hence the equation of Boyle's law to fit these data is *p v* = 24 lb.-ft. Since *p* is expressed in lb/ft^2 and *v* in ft^3, the constant 24 must have the units lb.-ft.

Parabola: A common physical relationship is of the type expressed by the equation

$$y = kx^2 \tag{5}$$

The plot of the data of this equation results in a parabola, shown by the solid line of Fig. 6. It is evident that Eq. 5 would plot as a straight line on rectangular coordinate paper providing the graph axes are expressed as y and as x^2. This is the straight dash line of Fig. 6.

Examples of physical relationships which have the form of Eq. 5 are: *Length vs. Period* of a simple pendulum, *Tension vs. Frequency* of a vibrating stretched string, and *Distance vs. Time* for the free fall of a body from rest. The free fall distance versus time equation was derived as Eq. 3 and for v_o = 0 is

$$s = \frac{1}{2}gt^2 \tag{6}$$

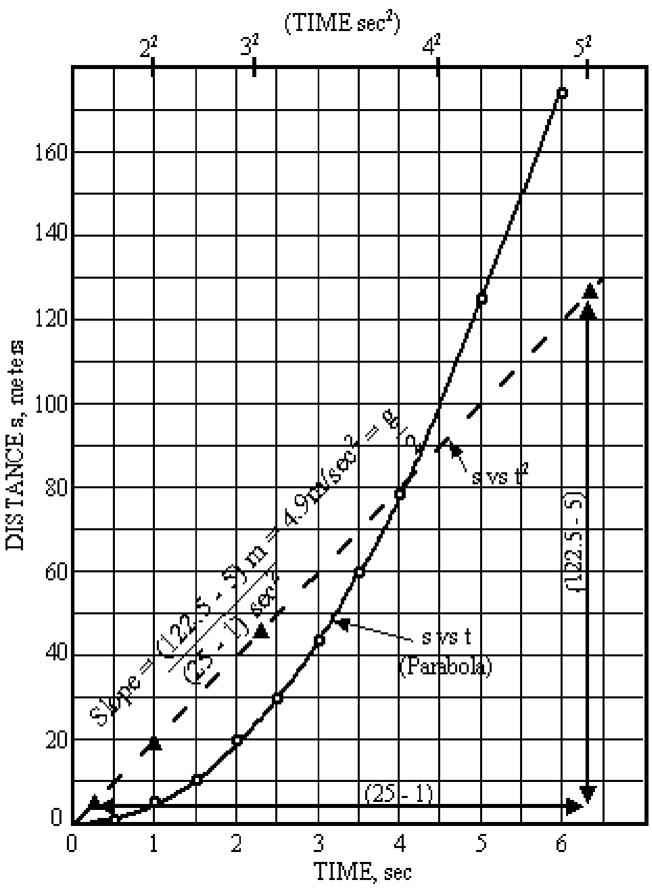

Fig. 6. The parabola, Distance-Time relation for a free-fall body.

The computed values t and s of Table II are plotted in Fig. 6. The use of the logarithm values is discussed in the next section.

Power law: Equation (6), $s = \frac{1}{2}gt^2$, is an example of a power-law equation. These have the general form

$$y = kx^n \tag{7}$$

where the power n may be any value.

Table II

FREE FALL DISTANCE VS TIME, $s = \frac{1}{2}gt^2$

t (sec)	s (m)	$\log_{10}t$	$\log_{10}s$
0	0	----	----
0.5	1.23	$\bar{1}.699$	0.090
1.0	4.9	0.000	0.690
1.5	11.0	0.176	1.041
2.0	19.6	0.301	1.292
2.5	30.6	0.398	1.486
3.0	44.0	0.477	1.644
3.5	60.0	0.544	1.778
4.0	78.4	0.602	1.894
4.5	99.2	0.653	1.997
5.0	122.5	0.699	2.088

This equation expressed logarithmically to the base 10 becomes

$$\log y = \log k + n \log x$$

$$\log x = \left[\frac{1}{n}\right]\log y - \left[\frac{1}{n}\right]\log k \tag{8}$$

or

$$\log x = \left[\frac{1}{n}\right]\log y + c \tag{9}$$

Equation 8 (and also Eq. 9) is the equation of a straight line which is linear in log x vs. log y. It has a slope equal to l/n, where n is the power of x.

The numerical value of k must be that of y when x = 1 (see Eq. 7). This is also evident in Eq. 8 since when x = 1 the value of log x = 0 and since n ≠ 0,

$$\log k = +\log y$$

or

$$|k| = |y|$$

To test these conclusions a plot of the log values of Data Table II, obtained by using the equation

$$s = \frac{1}{2}gt^2$$

should give a straight line of slope $s = \frac{1}{2}gt^2$, since the power of t is 2. This plot is shown in Fig. 7. Note that it does give the correct slope value. The log value for t = 0.5 sec was not plotted. Since its value is 9.699-10, it could have been included as a value -0.301 on the y axis.

To eliminate the task of finding the logarithm values for the data the original readings of s and t may be plotted directly on *log-log* paper. This would be the usual procedure when the data are suspected of being a power-law type. Figure 8 is a plot of t and s of Table II on two-cycle log-log paper.

Since the length of the cycle, 1 to 10 or 10 to 100 etc., is the same for both axes, the slope of the line is obtained from

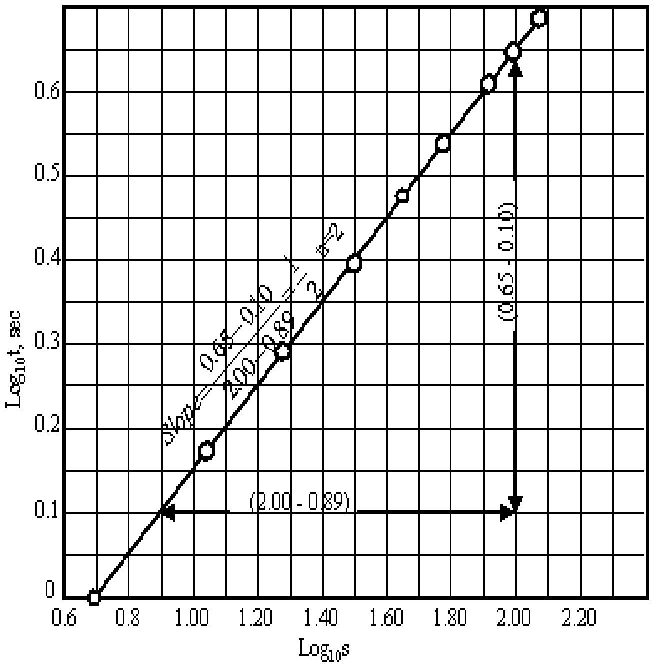

Fig. 7. $Log_{10}t$ vs $Log_{10}s$ for free fall.

actual *length measurements*, as shown on Fig. 8. If the cycle lengths on the *x* and *y* axes are not the same, the "slope value" so obtained must be multiplied by a factor which is the ratio of the cycle lengths. The slope of Fig. 8 is again 1/2, showing that the power *n* of *t* is 2, that is $s = kt^2$.

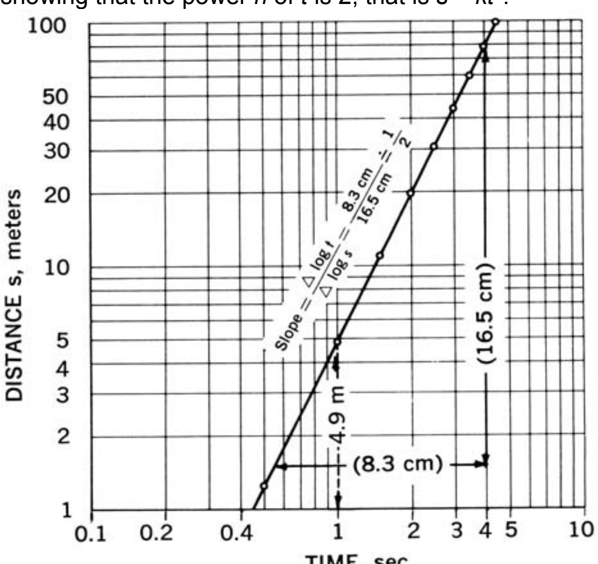

Fig. 8. Power curve, $s = kt^2$, on log log paper.

The constant *k* of Eq. 7 is the length reading for *t* = 1 sec, see dash line Fig. 8, giving again, as it must, the equation for the data of Table II

$$s = (4.9 m/\sec^2)t^2 = \frac{1}{2}gt^2$$

Figure 9 has samples of several power curves. Their equations are given in Table III. What letter designation fits each of the open sections? The constant *k* is the *y*-axis

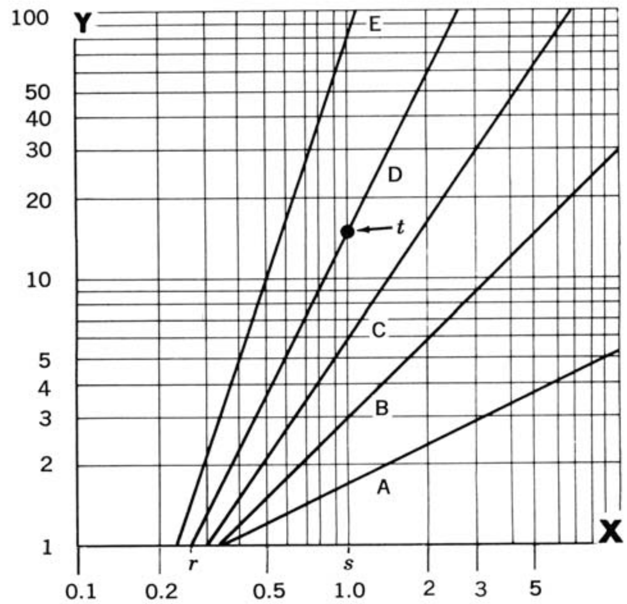

Fig. 9. Illustrative samples of power curves.

value for *x* = 1, and the power is the length ratio of ordinate to abscissa lengths of a cycle.

Table III.

EQUATIONS OF POWER CURVES.

Equation	Curve
$y = 15x^2$	D
$y = 3x$	
$y = 80x^2$	
$y = 6x^{1.5}$	
$y = 1.7\sqrt{x}$	

For example, in curve *D*, *k* = 15 because *y* = 15 when *x* = 1; and since $\frac{st}{rs} = 2$, the power of *x* = 2.

Exponential law: A good example of the exponential law relationship is evident in the ideal equation developed for thermal conductivity. See Selective Experiments in Physics Nos. H52b and H52c. This equation is used in the laboratory to determine the conductivity constant *K* expressed as

$$t = \left[\frac{lms}{KA}\right](\log_e I - \log_e I_o) \qquad (10)$$

where I is a galvanometer reading.

This discussion centers (a) on transforming Eq. 10 so that the data plotted on *semilog* paper will give a straight-line curve and (b) on the method of determining the slope value on this type of graph paper. Since the $\log_e 10 = 2.303$, expressing the equation in logarithms to base 10 gives

$$t = -\left[\frac{2.303 lms}{KA}\right](\log I - \log I_o)$$

Hence

$$\log I = \log I_o - \left[\frac{KA}{2.303lms}\right]t \qquad (11)$$

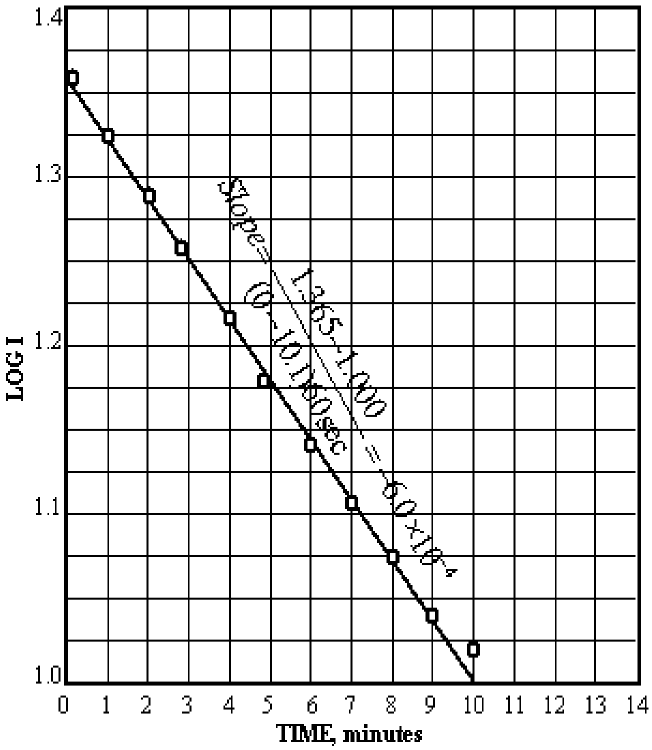

Fig. 10. Graph of log I vs time.

Equation 11 is a linear equation which will graph as a straight line when log I is plotted against *t*. The slope of the line is

$$\frac{\log I}{t} = -\left[\frac{KA}{2.303lms}\right]$$

Two graphs are possible: (a) the plot may be on rectangular coordinate paper by plotting Log *I* against *t*; (b) the values of *I* and *t* may be plotted directly on paper having a logarithm scale for *I* and a uniform scale for values of *t*. Such paper is called *semilog paper*. The two graphs are shown in Fig. 10 and Fig. 11, using the data of Table IV. The two graphs must give the same slope, thus providing a check on the method used to find the slope of a line on semilog paper. Remember that on logarithm paper the logarithms of the numbers are proportional to the distances from the *origin of the cycle* just as is true for slide rule scales.

The graph Fig. 10 on rectangular coordinate paper gives a slope value of -6.0×10^{-4} per sec. Thus, since the slope $= -\dfrac{KA}{2.303lms}$ the conductivity constant *K* of the test material used may now be computed provided it is the only unknown quantity of the right hand member.

The graph of Fig. 11 is on semilog paper. The abscissa for time values is a uniform scale, whereas the ordinate is a log scale. In this case only one cycle is used. The curve is a

Table IV.

DATA: THERMAL CONDUCTIVITY EXPERIMENT.

t (min)	*I* (galvanometer)	log*I*
0	23.2	1.365
1	21.1	1.324
2	19.5	1.290
3	18.0	1.255
4	16.4	1.215
5	15.0	1.176
6	13.8	1.140
7	12.7	1.104
8	11.9	1.076
9	11.0	1.041
10	10.4	1.017

straight-line. To find its slope value consider the following. Suppose in Fig. 11 that the measured length *ab* of one cycle is 22.4cm. For the triangle *ecd* the slope of the line *ed* on linear graph paper would be *ce/cd*. However in this case one needs the ratio of log *I/t*, and since the logarithms are proportional to the distances, the constant of proportionality being the length of 1 cycle, or 22.4cm.

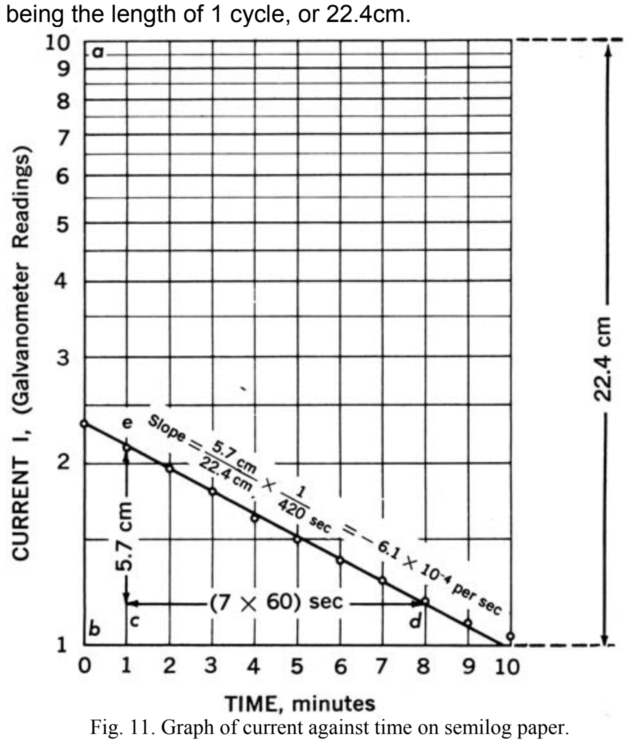

Fig. 11. Graph of current against time on semilog paper.

Thus the slope of *ed* on semilog paper is

$$\frac{ce/ab}{cd} = \frac{-5.7/22.4}{7 \times 60} = -6.1 \times 10^{-4} \text{ per sec.}$$

This is the same slope, within experimental error, obtained

from the graph of Fig. 10.
Another method to obtain the slope of the line ed of Fig. 11 is as follows. The slope $a = \dfrac{\Delta \log I}{\Delta t}$

EXERCISES AND QUESTIONS

1. A meter, designed to read luminous flux, gives the following data for a hanging lamp. The directions below and above the lamp are 0° and 180° respectively. All readings were taken at the same distance from the lamp.

Angle	(Divisions) Meter Reading	Angle	Meter Reading
0°	90	60°	50
10	85	75	36
15	78	90	33
20	68	110	29
25	57	125	25
30	52	140	18
40	55	160	14
50	58	180	14

Plot the data and interpret the curve. Sketch a lamp arrangement which would give these data.

2. The transmission curve of a given glass light filter is shown in Fig. 12. The letters V B G Y O R are color designations, Violet to Red. Interpret the curve giving wavelengths of maximum and minimum transmissions.
What is the color of the light transmitted through this filter when the light source is
(a) a sodium lamp which emits yellow light of wavelength 580mμ?
(b) a white light comparable to the sun?

3. The following data were obtained in an experiment relating the independent variable *i* to the dependent variable *d*.

i	d
0.5	0.2
1.0	3.1
1.5	6.2
2.0	9.3
2.5	12.2
3.0	15.2

Plot these data on rectangular coordinate paper.
(a) What is the slope value?
(b) What is the value of the d intercept?
(c) Write the equation for the curve.
(d) What is the value of d when $i = 3.7$?

4. Given a table of data, what would be a quick mathematical method to show that the curve plotted on rectangular coordinate paper would be a hyperbola?

5. Suppose there is reason to believe that the following data support some form of power curve $y = ax^n$. Show how to proceed with the data to obtain the equation. Write the equation.

x	y
0.2	1.5
0.4	4.5
0.6	8.4
0.8	12.5
1.0	17.8
1.5	32.0
2.0	53.0
2.5	73.8
3.0	99.0

6. Would a plot of the values of *i* and *d* taken from the curve of Problem 3 plot as a straight line on log log paper? Why or why not? (If in doubt, try it.)

7. A rather common laboratory experiment gives these typical data.

x cm	y cm
1	7.5
2	10.8
3	13.3
4	14.2
5	13.5
6	11.4
7	8.6
8	5.5
9	2.9
10	1.3
11	1.2
12	2.8
13	5.9
14	9.3
15	12.4

Plot the curve on rectangular coordinate paper. What is the curve called? Write the equation for this curve when the line of the *x* axis gives symmetry to the curve.

8. Recheck the answers expressed in Table III. Construct another line of different power and write its equation.

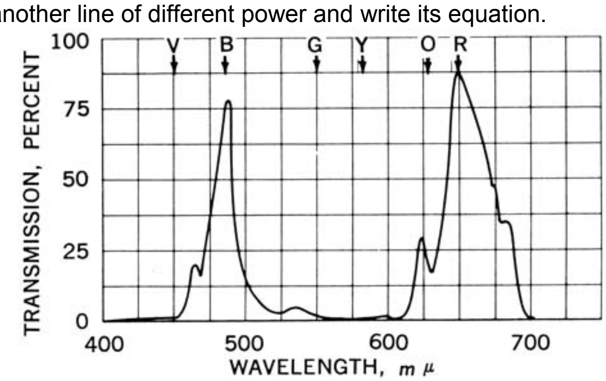

Fig. 12. Transmission of light by a color filter.