CENCO Physics — Selective Experiments In Physics

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ERRORS

I. INTRODUCTION: Observations are taken in the laboratory and from these observations certain conclusions are drawn. Since no observation or series of observations is absolutely accurate, it is often desirable to check the dependability of the conclusions by a study of the errors in the experiment.

Suppose that an experiment on the relation between the pressure and volume of a gas is performed in the laboratory and that the conclusion is the statement of the law that the volume is inversely proportional to the pressure. The experiment does not prove that the law is absolutely accurate but only that within certain limits, determined by the accuracy of the experiment, it has been found to be true. Small departures from the law will always be found and it should be possible to determine whether these departures indicate that the law is not exactly true or whether they are due to unavoidable experimental errors. Even if in this experiment no significant departures were found, observations with more refined apparatus might show conclusively that the law was only an approximation to the truth.

II. SIGNIFICANT FIGURES: In most experiments a detailed study of the probable error is not required. Usually it is sufficient to indicate roughly how accurate the result is. In elementary work all the sure figures and one (but only one) of the estimated figures are recorded so that in merely writing down an observation an estimate of its accuracy is indicated. In Table I are recorded five observations each of the length L, width W, and thickness T, of a block of wood. The first observation of T is 3.57cm. The first two figures are known but the third figure 7 is doubtful. Although the 7 is doubtful it does have significance. We feel reasonably sure that the correct value is between 5 and 9, say, and we have, therefore, three significant figures. The location of the decimal point has nothing to do with the number of significant figures. Whether written as 3.57cm, 35.7mm, or 0.0357m, this item has three significant figures. When a zero serves merely to locate the decimal point it is not a significant figure. However the zero in the third observation of W is the first doubtful figure and is significant. To omit this zero would be wrong, for that would indicate that the preceding 2 was doubtful.

TABLE 1
Measurements of the dimensions of a wooden block.

Obs. No.	length L	width W	thickness T
1	10.78 cm	8.21 cm	3.57 cm
2	10.80	8.22	3.52
3	10.75	8.20	3.58
4	10.73	8.21	3.53
5	10.78	8.22	3.55
Av.	10.77	8.212	3.55

Each value of *T* has three significant figures and since there is considerable variation in the third figures, three significant figures should be used in expressing the average. In other words, the second 5 in the average is doubtful and the average is known to three significant figures. However, in the observations of *W* the variations in the third figure are so small that the third figure 1 in the average can hardly be called doubtful and in the average we are justified in keeping four significant figures.

The product *LWT* is 313.9735020cm^3. However, this does not correctly represent the volume. It indicates that all the figures are known except the final zero, and, of course, this is far from true. Since an error of 1%, say. in anyone of the factors will cause a 1% error in the result, the volume cannot be determined to any greater degree of accuracy than the least accurate of the factors. Although it is difficult to make hard and fast rules about significant figures we may say that, in general, *in multiplication and division the result should have as many significant figures as the least accurate of the factors*. In some cases the answer should have one more significant figure than the least accurate of the factors. For example, in the equation 9.8 x 1.28 = 12.5, if the answer is to be as accurate as the least accurate of the factors, it must have three significant figures although the least accurate factor has only two. An inspection of the equation should make clear why this is true. The rule must be supplemented by the judgment of the experimenter. The volume of the wooden block should, therefore, be recorded as 314cm^3. Do not carry worthless figures through a series of computations only to discard them at the end. To save time keep only one or two doubtful figures throughout the computations. This will not affect the accuracy of the result. In addition and subtraction the case is entirely different. Suppose that a certain metal rod is 126.73cm long at 20˚C and that experiment shows that when heated to 100˚C the increase in length is 0.2138cm. The new length is 126.73cm + 0.2138cm = 126.94cm. Since the numbers to which the 3 and 8 are to be added are unknown (there is no reason to believe they are zeros) the sum is known to 2 decimal places only. From the above illustration it should be clear that when numbers are properly arranged in columns for a computation *in* *addition or subtraction, if any digit in a column of digits is unreliable, the answer in that column is likewise unreliable*.

III. CLASSIFICATIONS OF ERRORS: An error that tends to make an observation too high is called a positive error and one that makes it too low a negative error. Errors can be grouped in two general classes, systematic and random. *A systematic error is one that always produces an error of the same sign*, e. g., one that would tend to make all the observations of some one item too small, say. *A random error is one in which positive and negative errors are equally probable*. Systematic errors may be subdivided into three groups: instrumental, personal, and external. An instrumental error is an error caused by faulty or inaccurate apparatus. Personal errors are due to some peculiarity or bias of the observer. External errors are caused by external conditions (wind, temperature, humidity, vibration, etc.)
To understand better these various kinds of errors, let us study the errors in a particular experiment. In this experiment two observers, in an attempt to measure the velocity of sound, use a pistol and a 1/20sec stop watch to measure the time required for sound to travel between two points. The first observer fires the pistol and the second observer uses the stop watch to measure the time required for the sound to reach him. This simple experiment may be used to illustrate all the types of errors listed above. It should be possible to determine the sources of error. analyze these errors, and assign each to its proper classification. This is done in the paragraphs immediately following.

(A) **Systematic Errors**.

(1) ***Instrumental Error***. Since the stop watch is the only instrument used in this experiment, all instrumental errors must come from faults in the watch. A stop watch that did not run at the proper rate would cause an instrumental error. If the watch ran too fast all the observed times would be too high. Readings taken by a stopwatch are also subject to another type of systematic error. It takes an appreciable time merely to start or stop the mechanism of a watch. If the lag at starting is not equal to the lag at stopping, error will result. To reduce instrumental errors the instrument should be checked with an accurate standard and the necessary corrections applied to the observations. If a high degree of accuracy is called for, the instrument may be sent to the National Bureau of Standards for calibration.

(2) ***Personal Error***. There are several ways in which the bias of the observer or his particular method of taking data might produce systematic errors. Two of these will be considered. Owing to the difference in the way he reacts to the flash of the pistol and the report that he hears, an observer might tend always to get time readings which are too large, say. It also might be true that having been warned by the flash of the gun he will be more alert and react more quickly in stopping the watch than in starting it. Personal errors maybe minimized by taking the observations under various conditions and by using several observers working independently.

(3) ***External Errors***. External errors are usually caused by conditions over which the observer has no control. Therefore they cannot be eliminated, but necessary corrections may be applied. In this experiment the error due to wind might be quite serious. To keep this error small the experimenters might follow any of the following procedures: choose a time when there is very little wind, measure the wind velocity and make the necessary corrections, or change positions with each other so that when the results are averaged the errors tend to cancel out.

(B) **Random Errors**. In this experiment 26 observations of the time required for sound to travel between two points are taken. These observed times are recorded in column II of Table II. Since a systematic error would affect each observation in the same way, the variations in this column

TABLE II

Observations of the time t required for sound to travel between two points and the deviation d of these observations.

I	II	III		IV
Obs. No.	t in sec	d in sec		d^2 in sec^2
		negative	positive	
1	1.50	.040		.0016
2	1.65		.110	121
3	1.45	.090		81
4	1.50	.040		16
5	1.30	.240		576
6	1.40	.140		196
7	1.60		.060	36
8	1.65		.110	121
9	1.75		.210	441
10	1.55		.010	1
11	1.50	.040		16
12	1.60		.060	36
13	1.50	.040		16
14	1.45	.090		81
15	1.55		.010	1
16	1.40	.140		196
17	1.80		.260	676
18	1.45	.090		81
19	1.55		.010	1
20	1.65		.110	121
21	1.65		.110	121
22	1.50	.040		16
23	1.55		.010	1
24	1.45	.090		81
25	1.60		.060	36
26	1.50	.040		16
Sum	40.05	1.120	1.130	.3101
Av.	1.540	.0865		

are not due to systematic errors. We believe that each of these errors is due to a large number of factors each of which adds its own small contribution to the total error. Since these factors are unknown and variable it is assumed that the resulting error is a matter of chance and therefore positive and negative errors are equally probable. Such errors are called random errors; some authors prefer to call them accidental errors. Owing to the fact that random errors are subject to the laws of chance, their effect on the experiment maybe made quite small by taking a large number of observations. It should be clear why increasing the number of observations has no effect on systematic errors.

IV. PROBABLE ERROR: Since the variations in the observed times *t* (column II, Table II) are governed by chance, one may apply the laws of statistics to them and arrive at certain definite conclusions about the magnitude of the errors.

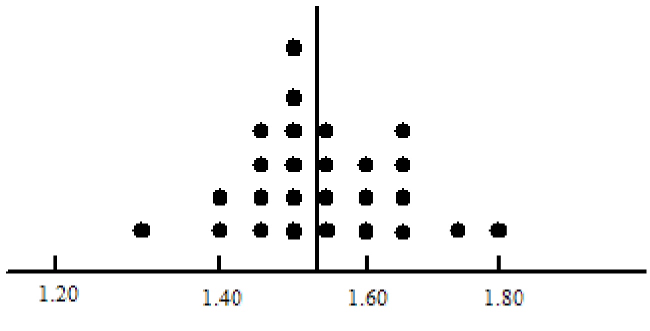

Fig. 1. Distributions of Observation

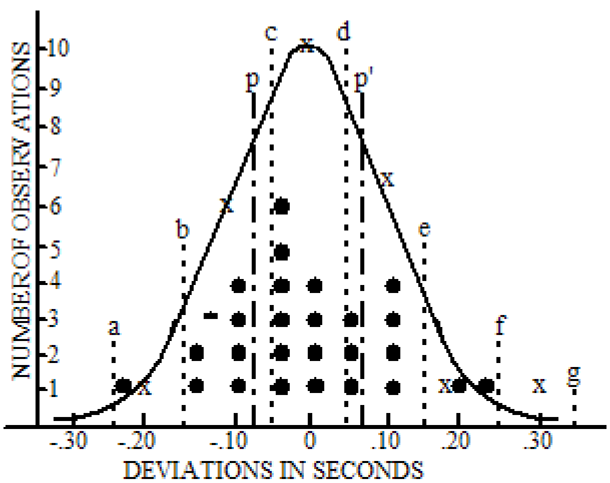

Fig. 2. ERROR CURVE. Shows the relation between the magnitude of the deviations and the frequency with which they occur.

No attempt will be made to derive these statistical laws, but the ones that are pertinent to this discussion will be simply stated. Along the horizontal axis of Fig. 1 are plotted observed times and each dot represents one observation. For example, three of the twenty-six observations of time gave 1.60sec. It is clear from this figure that the data tend to cluster about a certain mean value. What value is the one having the highest probability of being correct? To answer this question the methods of statistics are used and although the proof is rather difficult the conclusion is quite simple. It indicates that the best average is obtained by dividing the sum of the t's by the number of observations n. This is the simple method of averaging with which the reader is already familiar and an average obtained in this way is known as the arithmetic mean a.m. In other words, *the arithmetic mean, obtained by dividing the sum of the observed values by the number of observations taken, represents the best value obtainable from a series of observations.* The *a.m.* in this experiment is found to be 1.540sec and is represented by the vertical line in Fig. 1.

The difference between an observation and the arithmetic mean a.m. is called the deviation d and the average deviation *a.d.* is a measure of the accuracy of the experiment. Obviously, *the average deviation is the sum of the deviations divided by the number of observations, a.d. = Σd/n*. Deviations of the observations from the a.m. are recorded in column III, in which negative deviations are placed on the left side of the column and positive deviations on the right. Adding separately it is seen that the sum of the positive deviations is approximately equal to the sum of the negative deviations. Ideally they should be exactly equal.

To study the distribution and significance of the deviations, these deviations are plotted in Fig. 2 in much the same way that observations were plotted in Fig. 1. Each dot represents the deviation of one observation. Let us divide the observations into groups by the vertical lines a, b. c, etc., which divide the figure into slices each 1/10 sec wide with zero deviation at the midpoint of the central slice. In the figure the number of observations in each slice is represented by across (X) at the midpoint of the slice and the best smooth curve is drawn through these points. The curve therefore represents the relation between the magnitude of the deviations and the frequency with which they occur. From this graph the following general rules may be inferred:

(1) *Positive and negative deviations are equally probable.*

(2) *Small deviations occur more frequently than large deviations.*

Theory indicates that the relation between the probable error *p.e.* of a single observation, the sum of the deviations *Σd* (added without regard to sign) and the number n of observations is given by the equation

$$p.e. = 0.8453\frac{\sum d}{\sqrt{n(n-1)}} \qquad (1)$$

If this equation is applied to the data in Table II, we find that

$$p.e. = \frac{0.8453 \times 2.25}{\sqrt{26 \times 25}} = 0.075 \qquad (2)$$

This does not mean that no observation will deviate by more than 0.075 sec from the mean. It does mean that the chances are 50 to 50 that the error of a single observation will not exceed 0.075 sec; or. what amounts to the same thing, if a large number of observations are taken, half of these observations will have errors less than this amount. In Fig; 2 vertical lines P and P' are drawn at d = -0.075 and d = +0.075. It can be seen that halt of the observations lie within these limits. Obviously, the probable error *P.E.* of the *a.m.* is less than the probable error *p.e.* of a single observation. P.E. may be computed by the equation

$$P.E. = 0.8453\frac{\sum d}{n\sqrt{n-1}} = 0.8453\frac{a.d.}{\sqrt{n-1}} \qquad (3)$$

remembering that *a.d.* = *Σd/n*. which in this experiment gives

$$P.E. = \frac{0.8453 \times 2.25}{26 \times 5} = 0.015 \tag{4}$$

and one may write as the result of the twenty-six observations of time t that t = 1.540 ± 0.015. Again this does not mean that one can be sure that the correct value is between 1.525 sec and 1.555 sec but that the chances are even that it lies between these limits.

Comparing Eq. (1) with Eq. (3) it is found that the probable error of the mean of n observations is $1/\sqrt{n}$ times the probable error of a single observation. $P.E. = p.e./\sqrt{n}$. For example, the result obtained by averaging 9 observations is three times as reliable as a single observation and 81 observations are three times as reliable as 9. Since the accuracy increases as the square root of the number of observations taken, it is evident that an observer is not justified in spending the time required to take a very large number of observations. For most experiments 5 or 10 observations should be sufficient.

V. PROBABLE ERROR-ANOTHER METHOD: The probable error may also be computed from the sum of the squares of the deviations Σd^2. The squares of the deviations are given in column IV, Table II. This method is more tedious and only slightly more accurate than the method discussed in Section IV. The equations used in this method and the numerical results for this experiment are given below. where the symbols have the same significance they had in Eqs. (1). and (3).

$$p.e. = 0.6745\sqrt{\frac{\sum d^2}{n-1}} = 0.6745\sqrt{\frac{0.3101}{25}} = 0.074 \tag{5}$$

$$P.E. = 0.6745\sqrt{\frac{\sum d^2}{n(n-1)}} = 0.6745\sqrt{\frac{0.3101}{26 \times 25}} = 0.015 \tag{6}$$

It is seen that this method gives substantially the same probable error as the one used in Section IV.

It was stated in Section IV that the best average of a series of observations is the a.m. It can also be shown that the best average is the value which makes Σd^2 a minimum. For this reason the branch of mathematics which has been employed in the study of errors is often called *The Method of Least Squares*.

VI. PERCENTAGE ERROR: In a great many cases one is not so much interested in the numerical error as in the percent of error. For example, in Section IV it was found that the probable error of a single observation is 0.075 sec. It is often desirable to express the error in percent of the thing being measured. The probable percentage error *p.p.e*. of a single observation is $\frac{.075}{1.54}$ 100% or 4.9%; the odds are even that an observation will not deviate more than 4.9% from the mean. Similarly, the probable percentage error *P.P.E*. of the mean is $\frac{.015}{1.54}$ 100% or .97%.

VII. PROPACATION OF ERRORS: So far this discussion has been limited to a study of the errors in a group of observations all measuring the same thing, namely, the time t required for sound to travel between two points. Since in this experiment the observers are interested in the velocity of sound, it will be necessary to measure the distance S between the two points and compute the velocity v from the measured values of t and S. What effect do errors in t and S have upon v? Can general laws be formulated governing the effect of errors in the several items on the computed result? We shall consider only two cases.

Addition and Subtraction. *The probable error of the result is the square root of the sum of the squares of the probable errors of the separate items*. For example, (12.15 ± 0.03)cm + (8.63 ± 0.04)cm - (6.15 ± 0.05)cm = (14.63 ± .07)cm, since $\sqrt{0.03^2 + 0.04^2 + 0.05^2} = 0.07$

Multiplication and Division. When the result is obtained by multiplication and division its probable percentage error is determined by the application of the following two rules:

(1) The P.P.E. of the result is the square root of the sum of the squares of the P.P.E.'s of the factors.

(2) In case a factor is raised to the nth power its P.P.E. should be multiplied by n.

For example, let us assume that the density D of a certain cylinder is computed from the equation $D = M/\pi r^2 h$ and that the *P.P.E.'s* of *M, r*, and *h* are 2%, 3%, and 5%, respectively. Then the P.P.E. of D is 8% since $\sqrt{2^2 + (2x3)^2 + 5^2} = 8$. From this it is clear that a 3% error in the radius r will more seriously affect the result than a 5% error in the height h. Having determined the *P.P.E*. of the result, the *P.E*. may readily be computed.

VIII. PROBABLE ERROR WHEN SYSTEMATIC ERROR IS PRESENT: In the discussion of probable error above only random errors were considered. Systematic errors will always be present and in some cases will be sufficiently large to affect the reliability of the result. The probable value of the systematic error may be determined from a separate experiment or, with an experienced observer, may be estimated. Calling the probable error of a single observation due to random errors r, the probable systematic error r_1, the probable error of the result due to both causes r_o, and the number of observations n, it can be shown that

$$r_o = \sqrt{\frac{r^2}{n} + r_1^2} \tag{7}$$

If r and r_1 cannot be expressed in the same units, probable

percentage errors must be used.
Eq. (7) shows that if n is large it is the systematic error that determines the reliability of the result and a very large number of observations would not be justified. In many experiments r_1 is small and only random errors need be considered.

IX. GENERAL: In the study of errors we have applied certain statistical laws to 26 observations. Actually these laws apply accurately only when the number of observations is quite large. It is unusual for 26 observations to agree as well with theory as the ones given in Table II. To those interested in a more rigorous and more complete treatment of errors the following books are recommended:

H. M. Goodwin, *Precision of Measurements and Graphical Methods*, McGraw-Hill, 1920;

D. Brunt, *The Combination of Observations*, Cam-bridge University Press, 1931;

G. V. Wendell and W. L. Severinghaus, *A Manual of Physical Measurements*, Privately Printed, 1918.

J. W. Mellor, *Higher Mathematics for Students of Chemistry and Physics*, Longmans Green & Co., 1929.