• C.J. Overbeck Northwestern University • R.R. Palmer Beloit College • Marsh W. White The Pennsylvania State University

# MODULUS OF RIGIDITY

**OBJECT**: To determine the modulus of rigidity of brass and steel.

**METHOD**: A long rod clamped at one end is twisted through a measured angle by a known torque. From the length and radius of the rod and the torque necessary to produce unit angle of twist (one radian), the rigidity modulus for the material of which the rod is made is calculated.

**THEORY**: All objects, when acted upon by forces, are deformed a certain amount. The magnitude of the deformation produced by a known applied force gives a measure of the elastic constant of the material. In the case of the modulus of rigidity it is convenient to consider a cube of material (Fig. 1) fixed at its lower face and acted upon by

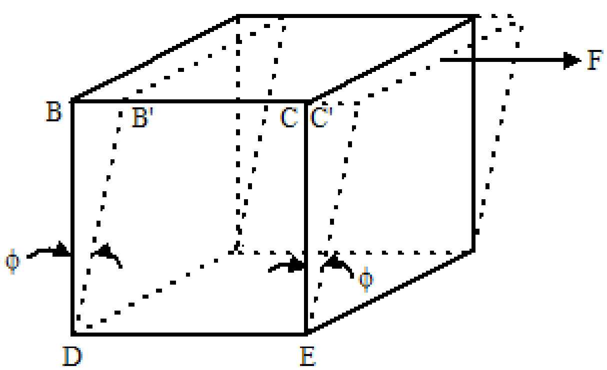


Fig. 1. Sharing of cubical block through angle $\phi$ by force F.

a tangential force $F$ at its upper face. This force causes the consecutive horizontal layers of the cube to be slightly displaced or sheared relative to one another. The stress producing this shear is the tangential force per unit area of the upper surface. Stress = $F/A$ where $F$ is the tangential force and $A$ the area of the surface upon which it acts. The strain produced in this cube by the stress is measured by the angle $\phi$, i.e., the ratio $\frac{BB'}{BD}$ if BB' is small compared with BD.

For stresses smaller than a certain limiting value, which depends on the material of the body to which they are applied, the strain disappears when the stresses are removed. If this limiting value is exceeded, the material is strained beyond its elastic limit and such strains are permanent; the material no longer returns to its original configuration when the stresses are removed. For stresses below this elastic limit of the material, Hooke showed that the strain produced is proportional to the applied stress, or

$$\frac{\text{stress}}{\text{strain}} = \text{constant}$$

For the shear type of strain the constant is called the modulus of rigidity $n$. Hence

$$n = \frac{F/A}{\phi} \tag{1}$$

In the c.g.s. system modulus of rigidity is measured in dynes per square centimeter.

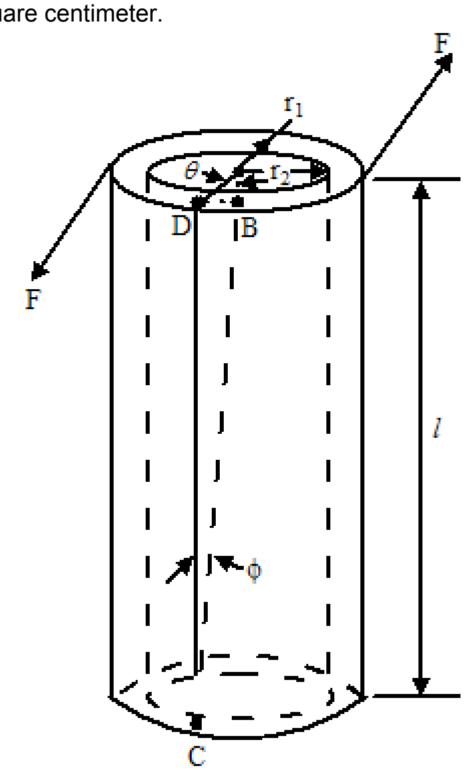


Fig. 2. Shearing of a cylinder by torque of moment F x r.

Consider a right circular hollow cylinder of height $DC = l$ and radii $r_1$ and $r_2$ (Fig. 2). Suppose the base of the cylinder is rigidly clamped while the upper face is subjected to a torque produced by the two tangential forces of magnitude $F$ shown in the diagram. This torque causes the upper face to be twisted through an angle $\theta$ and a vertical line such as DC to be shifted to BC through an angle $\phi$.

$$DB = r\theta = l\phi \tag{2}$$

where DB is the arc generated by the displacement of point D to B and *r* is the mean radius of the cross section of the ring. Each cross section of the cylinder is twisted or sheared slightly with respect to the section adjacent in such away that in a length *1* of the cylinder the angle of shear is $\phi$.

Let *f* be the shearing force over the area of cross section $A = \pi\left(r_1^{\,2} - r_2^{\,2}\right)$. From the expression for the modulus of rigidity *n*, Eq. (1),

$$n = \frac{f}{\pi\left(r_1^{\,2} - r_2^{\,2}\right)\phi} \tag{3}$$

or

$$f = n\pi\left(r_1^{\,2} - r_2^{\,2}\right)\phi = n\pi\left(r_1^{\,2} - r_2^{\,2}\right)\frac{r\theta}{l} \tag{4}$$

where the more easily measured angle $\theta$ has been substituted for $\phi$ by means of Eq. (2).

This shearing force *f* has a moment $f \times r$ about the axis of the cylinder so that

$$f \times r = n\pi\left(r_1^2 - r_2^2\right)\frac{r^2\theta}{l} \tag{5}$$

On the assumption that the average value $\dfrac{r_1^{\,2} + r_2^{\,2}}{2}$ is: equal to $r^2$

$$\begin{aligned} f \times r &= n\pi\left(r_1^{\,2} - r_2^{\,2}\right)\left(r_1^{\,2} + r_2^{\,2}\right)\frac{\theta}{2l} \\ &= \frac{n\pi\theta}{2l}\left(r_1^{\,4} - r_2^{\,4}\right) \end{aligned} \tag{6}$$

For a solid cylinder of outer radius *R*, which may be considered to consist of a number of hollow concentric cylinders having radii 0, $r_a$, $r_b$ . . . . . $r_{n-1}$, $r_n$, *R*, the total moment of the shearing force is the sum of a series of expressions similar to the right-hand side of Eq.(6)

$$\begin{aligned} \Sigma\left(f \times r\right) &= \frac{n\pi\theta}{2l}\left\{\left(r_a^{\,4} - 0^4\right) + \left(r_b^{\,4} - r_a^{\,4}\right) + \ldots\right. \\ &\left.\left(r_n^{\,4} - r_{n-1}^{\,4}\right) + \left(R^4 - r_n^{\,4}\right)\right\} = \frac{n\pi\theta R^4}{2l} \end{aligned} \tag{7}$$

But $\sum\left(f \times r\right)$is the total moment of the shearing forces about the axis of the solid cylinder and this must equal the moment *L* of the applied torque. Hence

$$L = \frac{n\pi\theta R^4}{2l}$$

or

$$n = \frac{2lL}{\pi\theta R^4} \tag{8}$$

The quantity L/$\theta$, the ratio of the moment of the applied torque to the angle of twist $\theta$ which it produces, is called the moment of torsion $L_o$.

$$L_o = L/\theta \tag{9}$$

$$n = \frac{2lL_o}{\pi R^4} \tag{10}$$

**APPARATUS**: The apparatus shown in Fig. 3 consists of a table clamp with ball-bearing hub to which is attached a wheel 15cm in diameter. A section of this wheel is graduated in degrees, enabling the twist to be measured up to 100°. The vernier arm is movable about the axis and the vernier graduations are such that readings may be made to 0.10. The hub of the wheel contains a socket with a single T-head

Fig. 3. Apparatus for determining the modulus of rigidity of a substance in the form of a cylindrical rod.

setscrew by means of which the rod is rigidly fastened in a center position. The other end of the rod is held in a similar socket mounted in another table clamp or "tail-stock." The rods, 100cm in length, are firmly attached at their ends to brass bushings having a deep longitudinal groove which fits into the socket of the table clamps. The graduated wheel has a fiat peripheral surface around which passes a steel ribbon carrying a weight holder. A set of half-kilogram weights ranging up to a maximum of 3kg are needed. Four rods, three of steel and one of brass, all 100cm in length but of various diameters, are provided. A micrometer caliper calibrated in centimeters, a pair of large calipers and a meter stick are also required.

**PROCEDURE**:

**Experimental**: The two table clamps must be so aligned that the large diameter steel rod remains straight when clamped in them. With the weight hanger on the end of the steel ribbon, move the vernier arm so that the vernier reads zero. Load the weight hanger until the rod is twisted about 90°.

Remove the added weights and notice whether the vernier again reads zero. If the vernier does not read zero, one or more of the following conditions may exist: (a) the rod is not firmly clamped into the end chucks; (b) there is undue friction; (c) the vernier arm has been moved; (d) the rod is slipping in its end bushings. Before proceeding with the experiment, see to it that the readings of the vernier before and after the weights are added are practically the same. Having determined the maximum weight necessary to produce a twist of about 90°, add successive weights of about one fifth the maximum and take the scale and vernier readings after adding each weight. Remove the weights in the same order in which they were added, again taking the scale and vernier readings. The two readings for the same weight should be practically the same though not necessarily identical. Measure the length $l$ of the rod between the brass end bushings. By means of a micrometer caliper determine the diameter of the rod in at least ten places along its length. This diameter must be carefully determined to obtain an accurate result for the modulus of rigidity. Measure the diameter of the pulley over which the steel ribbon passes. Record all the data in tabular form. Repeat the experiment on the two rods of brass and steel of the same diameter and also with the steel rod using two different lengths, 50 and 100cm.

**Analysis of Data**: 1. Plot each of the sets of data of load and angular twist, using the average of the two values of the angle. Draw the best straight line through the points, and from the slope determine the load necessary for a twist of 90°, i.e., $\pi/2$ radians. From this calculate $L_o$ for each set of data. Remember $L_o$, the moment of torsion, is numerically the moment of force expressed in centimeter-dynes required to produce a twist of *l* radian.

2. Using the appropriate values of the radius $R$ and the length $l$ of the rod, together with the value of $L_o$, calculate from Eq. (10) the value of the modulus of rigidity of the various rods. Express the results to the proper number of significant figures multiplied by 10 raised to the proper power.

3. For the steel rod having a clamp at its center, calculate the ratio of its moment of torsion to that when the full length of the rod is involved and compare this ratio with that of the two lengths used.

4. For two of the steel rods of different diameters, calculate the ratio of the moments of torsion and compare this result with the ratio of the radii raised to the fourth power.

**QUESTIONS**: 1. If an error of one half of one per cent is made in the determination of the radius of the rod, all other quantities being assumed accurate, what error will be introduced into the final value of $n$?

2. Which elastic modulus is involved in the bending of a beam?

3. If the graphs of the load and angular twist are not straight lines, does this necessarily imply that the elastic limits are exceeded? Draw a rough graph of load versus angular twist for the case of a rod in which the elastic limit has been exceeded.